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# Bäcklund Transformations and exact time-discretizations for Gaudin and related models

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**PhD THESIS**

*Ai miei genitori*

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# Chapter 1

# Introduction

The development of methods suitable to obtain numerical, approximate or exact solutions of non linear differential equations has shown an irregular evolution in the course of history of sciences. The key achievement, symbolizing the starting point of modern studies on the subject, can be considered the formulation of the *fundamental theorem of calculus*, dating back to the works of Isaac Barrow, Sir Isaac Newton and Gottfried Wilhelm Leibniz, even though restricted versions of the same theorem can be traced back to James Gregory and Pierre de Fermat. The power series machinery and a table of primitives compiled by himself, enabled Newton to solve the first remarkable integrable system of Classical Mechanics, that is the Kepler two-body problem. During the eighteenth century there was an enormous amount of mathematical works, often inspired by physical problems, on the theory of differential equations. We have to mention Lagrange and Euler, the leading figures in the development of theoretical mechanics of the time, and Gauss, that expanded the results on perturbations and small oscillations. In this century emerged a formalization for the theory of solutions, including methods by infinite series: these results were applied mainly to the theories of celestial mechanics and of continuous media. The rely on the systematic results that were found, lead Laplace to believe in a completely deterministic universe. In the subsequent years the theory was enriched with the existence and uniqueness theorems, and with the theorem of Liouville on the sufficient condition to integrate a dynamical system by quadratures. At the same time mathematicians understood the importance to view some differential equations just as a *definition* of new functions and their properties. In this contexts the works of Sophus Lie put the theory of evolution equations on more solid foundations, introducing the study of groups of diffeomorphisms, the Lie groups, in the field of differential equations: this made clear that the difficulties arising in finding the solution of differential equations by quadrature often can be brought back to a common origin, that is the joint invariance of the equations under the same infinitesimal transformation. Soon after Lie, Bäcklund and Bianchi, thanks also to a mutual influence one had on the others, established the foundations of the theory of surface transformations and of first order tangent transformations with their application to differential equation: in the next section I will give a short historical overview of the Bäcklund transformation theory just starting by the results of Bianchi, Bäcklund and Lie.

## 1.1 An overview of the classical treatment of surface transformations.

There exist several excellent books covering all the material reviewed in this section. This survey is based mostly on [18],[6],[101],[73],[54] [100].
As often happens in sciences history, researches in a new field can pose new queries but can also give unexpected answers to, at first sight, unrelated questions. So in the last 19th century, the Bäcklund transformations were introduced by geometers in the works on pseudospherical surfaces, that is surfaces of constant negative Gaussian curvature. This is a brief review of those results.
Consider a parametric representation of a surface $S$ in the three dimensional euclidean space: the coordinates of a point $\mathbf{r}$ on the surface are continuous and one-valued functions of two parameters, say $(u, v)$, so that

$$\mathbf{r} = \mathbf{r}(u, v).$$

If one considers a line on the surface, defined for example by a relation between $u$ and $v$ of the type $\phi(u, v) = 0$, then the infinitesimal arc length on this curve is defined by

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r}.$$

As a result of the (u,v) parametrization, this arc length can be rewritten also as:

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2, \tag{1.1}$$

where $E = \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial u}$, $F = \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v}$ and $G = \frac{\partial \mathbf{r}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial v}$. Since the curve is arbitrary, $ds$ is called the linear element of the surface and the quadratic differential form given by $Edu^2 + 2Fdudv + Gdv^2$ is called the *first fundamental form* of $S$. The values of $E$, $F$ and $G$ completely determine the curvature $K$ of the surface $S$, explicitly given by the following formula [18]:

$$K = \frac{1}{2H}\left(\frac{\partial}{\partial u}\Big(\frac{F}{EH}\frac{\partial E}{\partial v} - \frac{1}{H}\frac{\partial G}{\partial u}\Big) + \frac{\partial}{\partial v}\Big(\frac{2}{H}\frac{\partial F}{\partial u} - \frac{1}{H}\frac{\partial E}{\partial v} - \frac{F}{EH}\frac{\partial E}{\partial u}\Big)\right), \tag{1.2}$$

where for simplicity I have posed $H = \sqrt{EG - F^2}$. In the 19th century a question arose if it is possible to choose another parametrization of the surface, say in terms of

$(\alpha, \beta)$, so that the first fundamental form takes particular structures. More specifically it can be shown [18] that it is always possible to choose the parameters so that the first fundamental form is given by the special expression

$$d\alpha^2 + 2\cos(\omega) d\alpha d\beta + d\beta^2 \tag{1.3}$$

where $w(\alpha, \beta)$ is the angle between the parametric lines, i.e. the lines $r(\alpha_0, \beta)$ and $r(\alpha, \beta_0)$ where $\alpha_0$ and $\beta_0$ are two constant values.[1]. In terms of the parameters $(\alpha, \beta)$, by (1.2), the curvature $K$ is given by:

$$K = -\frac{1}{\sin(\omega)} \frac{\partial^2 \omega}{\partial\alpha\partial\beta}. \tag{1.4}$$

The pseudospherical surfaces are those of constant negative curvature. Let me take for simplicity the ray of such pseudospherical surfaces equal to 1, so that $K = -1$. For such surfaces it is possible to show [18] that, taking the asymptotic lines as coordinate lines $(\alpha, \beta)$, the first fundamental form is given by (1.3), so that, taking into account (1.4), the correspondent angle $\omega$ is a solution of the sine Gordon equation:

$$\sin(\omega) = \frac{\partial^2 \omega}{\partial\alpha\partial\beta}. \tag{1.5}$$

Conversely, at every solution of the sine Gordon equation, it corresponds a pseudospherical surface implicitly defined by the particular solution itself. In 1879, with purely geometric arguments, Luigi Bianchi showed [19] that given a pseudospherical surface and then a solution of sine Gordon equation it is possible to pass to another pseudospherical surface, that is to another solution of the sine Gordon equation. The Bianchi transformation linking two solutions of this equation reads:

$$\frac{\partial}{\partial\alpha}\left(\frac{\omega' - \omega}{2}\right) = \sin\left(\frac{\omega' + \omega}{2}\right), \tag{1.6}$$

$$\frac{\partial}{\partial\beta}\left(\frac{\omega' + \omega}{2}\right) = \sin\left(\frac{\omega' - \omega}{2}\right). \tag{1.7}$$

It is also possible to find the explicit expression of the transformed surface. In fact, if $\mathbf{r}$ and $\mathbf{r}'$ are respectively the position vectors of the pseudospherical surfaces corresponding to $\omega$ and $\omega'$, the transformation linking $\mathbf{r}'$ with $\mathbf{r}$ is [18]:

$$\mathbf{r}' = \mathbf{r} + \frac{1}{\sin(\omega)}\left(\sin\left(\frac{\omega - \omega'}{2}\right)\frac{\partial\mathbf{r}}{\partial\alpha} + \sin\left(\frac{\omega + \omega'}{2}\right)\frac{\partial\mathbf{r}}{\partial\beta}\right).$$

[1] The positive orientation of a line is given by the increasing direction of the non constant parameter, the angle is that between 0 and $\pi$ of the positive direction of the parametric lines [18].

By a direct inspection it is possible to see that the tangent planes at corresponding points of the two surfaces $S$ and $S'$ are orthogonal. In fact, if $\mathbf{N}$ and $\mathbf{N}'$ are the two unit vectors normal to $S$ and $S'$, then parallel to the vectors $\frac{\partial \mathbf{r}}{\partial \alpha} \wedge \frac{\partial \mathbf{r}}{\partial \beta}$ and $\frac{\partial \mathbf{r}'}{\partial \alpha} \wedge \frac{\partial \mathbf{r}'}{\partial \beta}$, the scalar product $\mathbf{N} \cdot \mathbf{N}'$ gives zero. In 1883 Bäcklund [22] successfully generalized the Bianchi construction letting the tangent planes of the two surfaces to meet at constant angle $\theta$ at corresponding points. This led to a one parameter family of transformations, the parameter being $a = \tan(\frac{\theta}{2})$. Explicitly the Bäcklund transformations on the two solutions of the sine Gordon equation read:

$$\frac{\partial}{\partial \alpha}\left(\frac{\omega' - \omega}{2}\right) = a \sin\left(\frac{\omega' + \omega}{2}\right), \tag{1.8}$$

$$\frac{\partial}{\partial \beta}\left(\frac{\omega' + \omega}{2}\right) = \frac{1}{a} \sin\left(\frac{\omega' - \omega}{2}\right), \tag{1.9}$$

while the transformations linking the two position vectors are:

$$\mathbf{r}' = \mathbf{r} + \frac{2a}{(a^2+1)\sin(\omega)}\left(\sin\left(\frac{\omega - \omega'}{2}\right)\frac{\partial \mathbf{r}}{\partial \alpha} + \sin\left(\frac{\omega + \omega'}{2}\right)\frac{\partial \mathbf{r}}{\partial \beta}\right). \tag{1.10}$$

Soon after this construction Lie [74] observed that the Bäcklund transformations can be indeed obtained by a conjugation of a simple Lie group invariance of the sine Gordon equation with the Bianchi transformation. The sine Gordon equation in fact is invariant under the scaling $(\tilde{\alpha} = a\alpha, \tilde{\beta} = \frac{\beta}{a})$, so that we can pass from the solution $\omega(\alpha, \beta)$ to the solution $\Omega(\alpha, \beta) = \omega(a\alpha, \frac{\beta}{a})$. The two solutions $\Omega$ and $\Omega'$, where $\Omega'$ is the Lie transformed of $\omega'$, are obviously linked by the Bäcklund transformations if $\omega$ and $\omega'$ are related by the Bianchi transformation:

$$\frac{\partial}{\partial \alpha}\left(\frac{\Omega' - \Omega}{2}\right) = a \sin\left(\frac{\Omega' + \Omega}{2}\right),$$
$$\frac{\partial}{\partial \beta}\left(\frac{\Omega' + \Omega}{2}\right) = \frac{1}{a} \sin\left(\frac{\Omega' - \Omega}{2}\right).$$

The process to pass from $\Omega$ to $\Omega'$ with the Bäcklund transformation $B_a$ can be then decomposed in this way: 1) pass from $\Omega$ to $\omega$ with the inverse of a Lie transformation $L^{-1}$; 2) pass from $\omega$ to $\omega'$ with a Bianchi transformation $B_{\frac{\pi}{2}}$; 3) pass from $\omega'$ to $\Omega'$ with a Lie transformation $L$. Formally:

$$B_a = L B_{\frac{\pi}{2}} L^{-1}.$$

In 1892 Bianchi [20] derived a non linear superposition principle for the solutions of the sine Gordon equation, the so called *Bianchi permutability theorem*. The question asked by Bianchi is simple: if $\omega_a$ is the solution of the sine Gordon equation obtained from $\omega$ with the Bäcklund transformation $B_a$ with parameter $a$, and $\omega_b$ is the solution obtained from $\omega$ with $B_b$, the Bäcklund transformation with parameter $b$, under what circumstances, by acting on $\omega_a$ with $B_b$ and on $\omega_b$ with $B_a$, it is possible to have $\omega_{ab} = \omega_{ba}$? The answer led to an algebraic expression of $\Omega = \omega_{ab} = \omega_{ba}$ in terms of $\omega$, $\omega_a$ and $\omega_b$. Following Bianchi [18], by using (1.8) one has:

$$\frac{\partial}{\partial\alpha}\left(\frac{\omega_a - \omega}{2}\right) = a\sin\left(\frac{\omega_a + \omega}{2}\right) \qquad \frac{\partial}{\partial\alpha}\left(\frac{\omega_b - \omega}{2}\right) = b\sin\left(\frac{\omega_b + \omega}{2}\right), \tag{1.11}$$

$$\frac{\partial}{\partial\alpha}\left(\frac{\omega_{ab} - \omega_a}{2}\right) = b\sin\left(\frac{\omega_{ab} + \omega_a}{2}\right) \qquad \frac{\partial}{\partial\alpha}\left(\frac{\omega_{ba} - \omega_b}{2}\right) = a\sin\left(\frac{\omega_{ba} + \omega_b}{2}\right). \tag{1.12}$$

By posing $\omega_{ab} = \omega_{ba} = \Omega$ and subtracting the two expressions for $\frac{\partial\Omega}{\partial\alpha}$ in (1.12), one easily obtains:

$$\frac{\partial\omega_a}{\partial\alpha} - \frac{\partial\omega_b}{\partial\alpha} = 2a\sin\left(\frac{\Omega + \omega_b}{2}\right) - 2b\sin\left(\frac{\Omega + \omega_a}{2}\right).$$

Introducing the other two expressions (1.11) in this equation one has:

$$a\sin\left(\frac{\omega_a + \omega}{2}\right) - b\sin\left(\frac{\omega_b + \omega}{2}\right) = a\sin\left(\frac{\Omega + \omega_b}{2}\right) - b\sin\left(\frac{\Omega + \omega_a}{2}\right).$$

This in turns implies:

$$a\sin\left(\frac{\omega - \Omega + (\omega_a - \omega_b)}{4}\right) = b\sin\left(\frac{\omega - \Omega - (\omega_a - \omega_b)}{4}\right).$$

By using the addition and subtraction formulae for the sin function, we obtain the relation known as the permutability theorem:

$$\tan\left(\frac{\Omega - \omega}{4}\right) = \frac{a+b}{a-b}\tan\left(\frac{\omega_b - \omega_a}{4}\right). \tag{1.13}$$

Note that one reaches the same result by starting from the expression of the Bäcklund transformation containing the $\beta$ derivative (1.9). At this point it is possible to construct, only with algebraic procedures, new pseudospherical surface from a given one. It is logic to suppose that the simplest of solutions of the sine Gordon equation has to correspond to the simplest of pseudospherical surfaces. A very simple family of pseudospherical surfaces are those of revolution. If the $z$ axis is the axis of rotation, the surface is fixed by the following parametrization [18]:

$$\mathbf{r} = (r\cos(\psi), r\sin(\psi), \phi(r)) \tag{1.14}$$

The parallels and meridians on the surface correspond respectively to the circles $r = const_1$ and the curves $\psi = const_2$. The first fundamental form (1.1) corresponding to this surface is:

$$ds^2 = \left(1 + \phi'(r)^2\right) dr^2 + r^2 d\psi^2 = d\tau^2 + r^2(\tau) d\psi^2,$$

having introduced the parameter $\tau$ given by $d\tau = \sqrt{(1 + \phi'(r)^2) dr^2}$. It is easy now to calculate the curvature of the surface with the formula (1.4). The result is:

$$K = -\frac{1}{r}\frac{d^2 r}{d\tau^2}. \tag{1.15}$$

The constraint $K = -1$ will give the surfaces of revolution with constant negative curvature determining the dependence of $r$ on $\tau$ and then fixing $\phi(r)$ thanks to the relation $d\tau = \sqrt{(1 + \phi'(r)^2) dr^2}$. The simplest solution of (1.15) is $r = e^\tau$. This gives for $\phi(r)$:

$$\phi(r) = \int \sqrt{1 - \left(\frac{dr}{d\tau}\right)^2} d\tau = \int \sqrt{1 - e^{2\tau}} d\tau$$

With the substitution $e^\tau = \sin(\eta)$, one has $\phi(\eta) = \int \frac{\cos(\eta)^2}{\sin(\eta)} d\eta$, so that $z = \phi(\eta) = cos(\eta) + \ln\left|\tan\frac{\eta}{2}\right|$. The surface, in terms of the parameters $\eta$ and $\psi$ is so given by:

$$\mathbf{r} = \left(\sin(\eta)\cos(\psi), \sin(\eta)\sin(\psi), \cos(\eta) + \ln\left|\tan\frac{\eta}{2}\right|\right) \tag{1.16}$$

and the corresponding first fundamental form is:

$$ds^2 = \frac{\cos(\eta)^2}{\sin(\eta)^2} d\eta^2 + \sin(\eta)^2 d\psi^2. \tag{1.17}$$

How detect what is the solution of the sine Gordon equation corresponding to this surface? Recall that the sine Gordon equation is written in the coordinates determined by asymptotic lines, so that one can try to write (1.16) in these coordinates. However it is simpler to parametrize both the sine Gordon equation and the surface (1.16) in terms of the "curvature coordinates", simply given by:

$$x = \alpha + \beta, \qquad y = \alpha - \beta.$$

In this frame the sine Gordon equation (1.5) becomes:

$$\omega_{xx} - \omega_{yy} = \sin(\omega) \tag{1.18}$$

and the first fundamental form corresponding to (1.3) is:

$$\cos^2\left(\frac{\omega}{2}\right) dx^2 + \sin^2\left(\frac{\omega}{2}\right) dy^2 \tag{1.19}$$

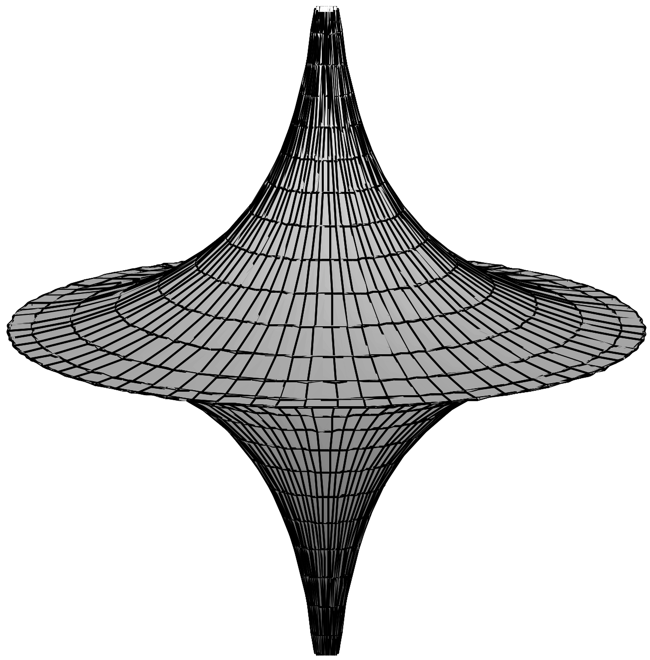

Figure 1.1: A Beltrami pseudosphere

Comparing this form with (1.17), one sees that, identifying $\psi = y$ and $\frac{d(\eta)}{\sin(\eta)} = dx$, they coincide if $\eta = \omega/2$. But, integrating $\frac{d(\eta)}{\sin(\eta)} = dx$, gives $\eta = 2\arctan(e^{x+c})$ where $c$ is the constant of integration, and then:

$$\omega = 4\arctan(e^{x+c}). \tag{1.20}$$

By a direct verification one sees that this is indeed a solution of (1.18). In the curvature coordinates the pseudospherical surface (1.16) reads:

$$\mathbf{r}(x, y) = \left(\frac{\cos(y)}{\cosh(x)}, \frac{\sin(y)}{\cosh(x)}, x - \tanh x\right). \tag{1.21}$$

This surface is known as the *Beltrami pseudosphere* [18], [100]. A plot is given in fig. (1.1). Now it is possible to obtain a ladder of pseudospherical surfaces and corresponding solutions of sine Gordon equation through the Bäcklund transformations. It is simpler to work in curvature coordinates; the Bäcklund transformations now read:

$$\frac{\partial}{\partial x}\left(\frac{\omega' - \omega}{2}\right) = \frac{1}{\sin(\theta)}\left(\sin\left(\frac{\omega'}{2}\right)\cos\left(\frac{\omega}{2}\right) - \cos(\theta)\cos\left(\frac{\omega'}{2}\right)\sin\left(\frac{\omega}{2}\right)\right) \tag{1.22}$$

$$\frac{\partial}{\partial y}\left(\frac{\omega'-\omega}{2}\right)=\frac{1}{\sin(\theta)}\left(\cos\left(\frac{\omega'}{2}\right)\sin\left(\frac{\omega}{2}\right)-\cos(\theta)\sin\left(\frac{\omega'}{2}\right)\cos\left(\frac{\omega}{2}\right)\right) \tag{1.23}$$

where I recall that $\theta$ is the angle at which the tangent planes of the two surface at corresponding points meet. Note that, if one starts with the solution $\omega = 0$, then by a simple integration it is readily obtained

$$\omega'(x,y)=4\arctan\left(e^{\frac{x-y\cos(\theta)}{\sin(\theta)}}\right), \tag{1.24}$$

that corresponds to (1.20) for $\theta = \frac{\pi}{2}$ (that is in the case of Bianchi transformations). To this solution one can verify that it corresponds a little modification of the Beltrami pseudosphere, that is the surface:

$$\mathbf{r}(x,y)=\left(\sin(\theta)\frac{\cos(y)}{\cosh(X)},\sin(\theta)\frac{\sin(y)}{\cosh(X)},\sin(\theta)\left(X-\tanh X\right)+y\cos(\theta)\right), \tag{1.25}$$

where $X = \frac{x-y\cos(\theta)}{\sin(\theta)}$. In terms of the variables $(X, y)$ it appears very similar to the surface of revolution (1.21). Indeed it is obtained by a rotation of the same curve that gives (1.21) plus a translation parallel to the same axis (the $z$-axis) in such a way that the ratio of the velocity of translation to that of rotation is a constant (given by $\cos(\theta)$). The surfaces obtained with this type of roto-translations are called *helicoids* [18]. Now it is possible to compose two solutions (1.24) and then use the Bäcklund transformations in the form (1.10) to obtain the corresponding pseudospherical surface. Let me use the parameters $\theta_1$ and $\theta_2$

$$\begin{cases}\omega_1=4\arctan\left(e^{\frac{x-y\cos(\theta_1)}{\sin(\theta_1)}}\right)=4\arctan\left(e^{X_1}\right),\\ \omega_2=4\arctan\left(e^{\frac{x-y\cos(\theta_2)}{\sin(\theta_2)}}\right)=4\arctan\left(e^{X_2}\right).\end{cases} \tag{1.26}$$

The composition of these two solutions, using the permutability theorem (1.13), gives:

$$\omega_{12}=4\arctan\left(\frac{\sin\left(\frac{\theta_1+\theta_2}{2}\right)\sinh\left(\frac{X_1-X_2}{2}\right)}{\sin\left(\frac{\theta_1-\theta_2}{2}\right)\cosh\left(\frac{X_1+X_2}{2}\right)}\right).$$

Corresponding to $\omega_1$ and $\omega_2$ one has the two surfaces:

$$\begin{cases}\mathbf{r}_1=\left(\sin(\theta_1)\frac{\cos(y)}{\cosh(X_1)},\frac{\sin(y)}{\cosh(X_1)},\sin(\theta_1)\left(X_1-\tanh X_1\right)+y\cos(\theta_1)\right)\\ \mathbf{r}_2=\left(\sin(\theta_2)\frac{\cos(y)}{\cosh(X_2)},\frac{\sin(y)}{\cosh(X_2)},\sin(\theta_1)\left(X_2-\tanh X_2\right)+y\cos(\theta_2)\right)\end{cases} \tag{1.27}$$

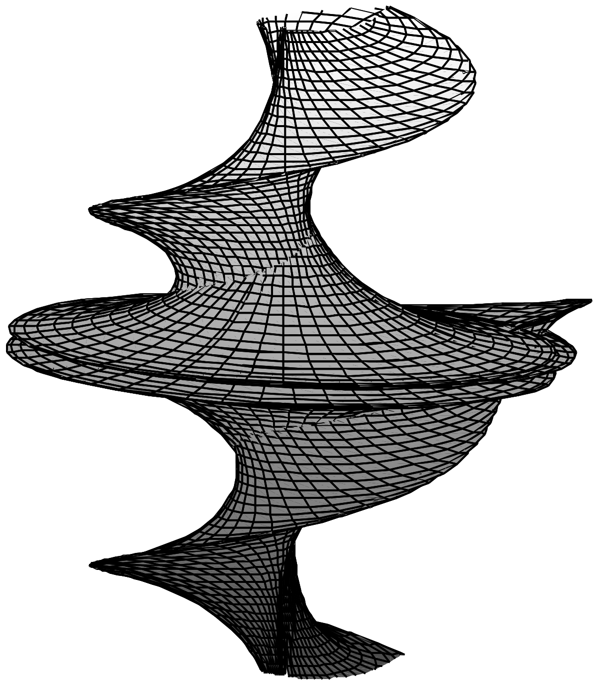

Figure 1.2: The two-soliton pseudospherical surface with $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = \frac{\pi}{2}$

In curvature coordinates the transformations (1.10) between $\mathbf{r}'$ and $\mathbf{r}$ is given by [18]:

$$\mathbf{r}' = \mathbf{r} + \sin(\theta) \left( \frac{\cos\left(\frac{\omega'}{2}\right)}{\cos\left(\frac{\omega}{2}\right)} \frac{\partial \mathbf{r}}{\partial x} - \frac{\sin\left(\frac{\omega'}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \frac{\partial \mathbf{r}}{\partial y} \right). \tag{1.28}$$

At this point there are all the elements to write down the explicit family of surfaces corresponding to the two-soliton solution $\omega_{12}$; by (1.28) it follows that:

$$\begin{aligned} \mathbf{r}_{12} =& \mathbf{r}_1 + \sin(\theta_2) \left( \frac{\cos\left(\frac{\omega_{12}}{2}\right)}{\cos\left(\frac{\omega_1}{2}\right)} \frac{\partial \mathbf{r_1}}{\partial x} - \frac{\sin\left(\frac{\omega_{12}}{2}\right)}{\sin\left(\frac{\omega_1}{2}\right)} \frac{\partial \mathbf{r_1}}{\partial y} \right) = \\ =& \mathbf{r}_2 + \sin(\theta_1) \left( \frac{\cos\left(\frac{\omega_{12}}{2}\right)}{\cos\left(\frac{\omega_2}{2}\right)} \frac{\partial \mathbf{r_2}}{\partial x} - \frac{\sin\left(\frac{\omega_{12}}{2}\right)}{\sin\left(\frac{\omega_2}{2}\right)} \frac{\partial \mathbf{r_2}}{\partial y} \right). \end{aligned} \tag{1.29}$$

A plot of a particular example of such surfaces is given in figure (1.2). The implication of the permutability theorem are noteworthy also by the point of view of dynamical systems. By its iteration it is in fact possible to construct $N$-soliton solutions (a non linear superposition of $N$ single soliton solutions) of the sine Gordon equation with a purely algebraic procedure. The procedure can be represented in a diagram known as

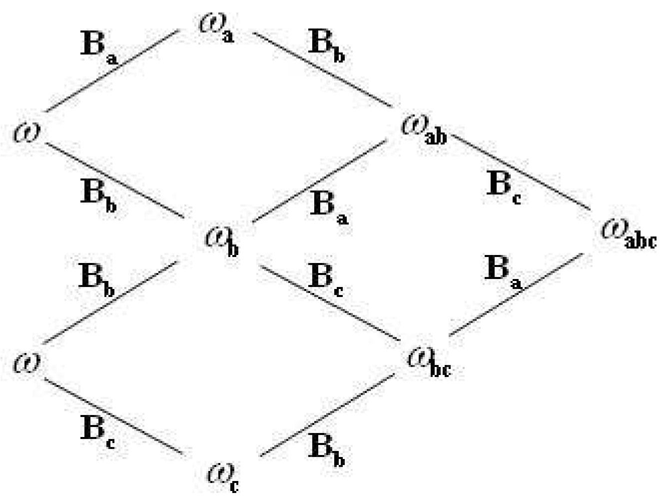


Figure 1.3: A Bianchi lattice

the *Bianchi lattice* (see figure (1.3)). As we will mention later, just a rediscovery of the permutability theorem in some physical applications allowed to rescue the subject of Bäcklund transformations in the second part of the twentieth century after the neglect in which it fell after the World War I. By the my point of view, the very deep coupling between algebraic and analytic results on solutions of non linear evolution equations on one hand and the geometry of surfaces on the other has been underestimated until today; yet the usefulness, that I hope will emerge also from this work, of the Bäcklund transformations in the theory of dynamical systems, but also as a tool for solving, numerically or analytically, systems of evolution equations, legitimates a broader interest in the geometrical aspects of such transformations.

## 1.2 The Clairin method

In 1903 Jean Clairin gave important contributions [29] to the subject of Bäcklund transformations. His results were broadly used in the 1970s. He had in mind to extend analytically the results of Bianchi to the case of a generic partial differential equation of second order. Although the Clairin approach is analytic and direct, often it requires tedious calculations. For completeness I will illustrate the method first with a simple generic situation and then getting again the Bäcklund transformations for the sine Gordon equation with an application of the method.
Suppose to have a generic partial differential equation of second order in two independent variables:

$$F(\alpha,\beta,\omega,\frac{\partial\omega}{\partial\alpha},\frac{\partial\omega}{\partial\beta},\frac{\partial^2\omega}{\partial\alpha^2},\frac{\partial^2\omega}{\partial\alpha\partial\beta},\frac{\partial^2\omega}{\partial\beta^2})=0. \tag{1.30}$$

Following Clairin [29], for simplicity of notation I pose:

$$p = \frac{\partial \omega}{\partial \alpha}, \quad q = \frac{\partial \omega}{\partial \beta}, \quad r = \frac{\partial^2 \omega}{\partial \alpha^2}, \quad s = \frac{\partial^2 \omega}{\partial \alpha \partial \beta}, \quad t = \frac{\partial^2 \omega}{\partial \beta^2}.$$

The notations for the transformed variables are the same, so $\tilde{p} = \frac{\partial \tilde{\omega}}{\partial \alpha}$ and so on. Clairin assumed that the first derivatives of $\omega$ are connected by the following system:

$$\begin{aligned} p &= f(\omega, \tilde{\omega}, \tilde{p}, \tilde{q}), \\ q &= g(\omega, \tilde{\omega}, \tilde{p}, \tilde{q}). \end{aligned} \tag{1.31}$$

The compatibility of this system requires

$$\frac{\partial p}{\partial \beta} = \frac{\partial q}{\partial \alpha}.$$

If this integrability condition is identically satisfied by equation (1.30) for the variable $\tilde{\omega}$, then the equations (1.31) are the Bäcklund transformations for (1.30). In fact, if one has a solution of (1.30), then the system (1.31) provides a new solution of the same equation by solving the resulting first order differential equations. At this point it is important to stress that, as noted by Forsyth [39] (see also [69]), when $f$ and $g$ in (1.31) are independent of $\tilde{\omega}$, then the compatibility equation, in some particular cases, can be seen as a *Lie contact transformation*. More specifically, when $\tilde{\omega}$ is absent, $\frac{\partial p}{\partial \beta} - \frac{\partial q}{\partial \alpha} = 0$ can be rewritten as:

$$\frac{\partial f}{\partial \omega} g - \frac{\partial g}{\partial \omega} f + \left( \frac{\partial f}{\partial \tilde{p}} - \frac{\partial g}{\partial \tilde{q}} \right) \tilde{s} + \frac{\partial f}{\partial \tilde{q}} \tilde{t} - \frac{\partial g}{\partial \tilde{p}} \tilde{r} = 0. \tag{1.32}$$

In order to satisfy this integrability condition, one can distinguish between two possibilities: or it is satisfied identically, so that the coefficient of $\tilde{s}$, $\tilde{r}$ and $\tilde{t}$ are zero and (1.32) can be transformed in a contact transformation [39]:

$$d\tilde{\omega} - \tilde{p} d\alpha - \tilde{q} d\beta = \mu(d\omega - p d\alpha - q d\beta),$$

or the integrability condition can be satisfied because $\tilde{\omega}$ is a solution of the partial differential equation (1.32): in this case one has a Bäcklund transformation. In order to clarify how practically works the method, consider again the sine Gordon equation:

$$\frac{\partial^2 \tilde{\omega}}{\partial \alpha \beta} = \sin(\tilde{\omega}).$$

For the sake of simplicity let me take equations (1.31) of the form:

$$\begin{aligned} q &= c(\tilde{\omega}) \tilde{q} + \mu(\omega, \tilde{\omega}), \\ p &= h(\tilde{\omega}) \tilde{p} + m(\omega, \tilde{\omega}). \end{aligned} \tag{1.33}$$

If the general form (1.31) for $p$ and $q$ is retained, one needs of a huger analysis but reaches the result given by (1.33). For more details see [6]. The compatibility condition (1.32) now reads:

$$\left(\frac{dc}{d\tilde{\omega}}-\frac{dh}{d\tilde{\omega}}\right)\tilde{q}\tilde{p}+(c-h)\sin(\tilde{\omega})+\frac{\partial\mu}{\partial\omega}p+\frac{\partial\mu}{\partial\tilde{\omega}}\tilde{p}-\frac{\partial m}{\partial\omega}q-\frac{\partial m}{\partial\tilde{\omega}}\tilde{q}=0. \tag{1.34}$$

Using (1.33), this relation becomes:

$$\left(\frac{dc}{d\tilde{\omega}}-\frac{dh}{d\tilde{\omega}}\right)\tilde{q}\tilde{p}+(c-h)\sin(\tilde{\omega})+\left(\frac{\partial\mu}{\partial\omega}d+\frac{\partial\mu}{\partial\tilde{\omega}}\right)\tilde{p}-\left(\frac{\partial m}{\partial\omega}c+\frac{\partial m}{\partial\tilde{\omega}}\right)\tilde{q}+\frac{\partial\mu}{\partial\omega}m-\frac{\partial m}{\partial\omega}\mu=0. \tag{1.35}$$

Differentiating with respect to $\tilde{p}$ and $\tilde{q}$ one sees that:

$$\frac{d}{d\tilde{\omega}}(c-d)=0.$$

Let me pose $c=-1$ and $h=1$. Differentiating (1.35) with respect to $\tilde{p}$ with these constraints one obtains:

$$\frac{\partial\mu}{\partial\omega}+\frac{\partial\mu}{\partial\tilde{\omega}}=0 \qquad \Longrightarrow \qquad \mu=\mu(\omega-\tilde{\omega}),$$

while, differentiating with respect to $\tilde{q}$

$$m=m(\omega+\tilde{\omega}).$$

Inserting this forms in (1.35), one is left with the functional differential equation:

$$-2\sin(\tilde{\omega})+\frac{\partial\mu(\omega-\tilde{\omega})}{\partial\omega}m(\omega+\tilde{\omega})-\frac{\partial m(\omega+\tilde{\omega})}{\partial\omega}\mu(\omega-\tilde{\omega})=0.$$

In order to solve this equation, let me differentiate with respect to $\omega$, getting:

$$\frac{\frac{\partial^2\mu(\omega-\tilde{\omega})}{\partial\omega^2}}{\mu(\omega-\tilde{\omega})}=\frac{\frac{\partial^2 m(\omega+\tilde{\omega})}{\partial\omega^2}}{m(\omega+\tilde{\omega})}.$$

The r.h.s. of this equation is a function of $\omega+\tilde{\omega}$ while the l.h.s. a function of $\omega-\tilde{\omega}$, so both sides must be equal to the same constant. Because in the functional differential equation a trigonometric function appears, this constant can be assumed real and negative, say $-K^2$. So:

$$\begin{aligned}\mu&=A\cos\left(K(\omega-\tilde{\omega})\right)+B\sin\left(K(\omega-\tilde{\omega})\right),\\ m&=C\cos\left(K(\omega+\tilde{\omega})\right)+D\sin\left(K(\omega+\tilde{\omega})\right).\end{aligned}$$

Substituting this forms in the functional differential equation and evaluating all at $\omega = 0$, one finds the constraints $2K = 1, AD = BC, AC + BD = 4$. The Bäcklund transformations for the sine Gordon equation are attained by posing $A = C = 0$, $\frac{D}{2} = \frac{2}{B} = a$. In this case the transformations (1.33) read:

$$\frac{q+\tilde{q}}{2} = \frac{1}{a}\sin\left(\frac{\omega-\tilde{\omega}}{2}\right),$$
$$\frac{p-\tilde{p}}{2} = a\sin\left(\frac{\omega+\tilde{\omega}}{2}\right),$$

that are exactly the relations (1.8) and (1.9). After the revival of the subject of Bäcklund transformations in the last half of the twentieth century, the construction of such transformations for a number of equations of physical interest (for example KdV, mKdV, NLS, Ernst equation) was obtained just using the Clairin method [6], [69], [66]. A last observation on the terminology usually found in the literature: commonly the transformation that links solutions of the same differential equation is called an auto-Bäcklund transformation, in opposition to the case of transformation linking solutions of two different differential equations: generally speaking this last one is the Bäcklund transformation. Since in this work I will deal only with auto-Bäcklund transformations, I will speak simply of Bäcklund transformations and no confusion can arise.

## 1.3 The *Renaissance* of Bäcklund transformations

After a nearly silent period in the scientific community on the subject, in 1953 the Bäcklund transformations and soliton theory took a new run to establish themselves in physics. About fifteen years before, Frenkel and Kontorova, in order to explain the mechanism of plastic deformations in the crystal lattice of the metals, introduced [40] a lattice dynamic model describing how many atoms can form long dislocation line. If $q_n$ is the distance of the $n$-th atom from its equilibrium position, $a$ is the lattice constant and $A$ and $B$ two constants, then the equations of motion for the $q_n$'s are:

$$m\frac{d^2q_n}{dt^2} = -2\pi A\sin\left(2\pi\frac{q_n}{a}\right) + B\left(q_{n+1} - 2q_n + q_{n-1}\right)$$

This is clearly the spatial discrete version of the sine Gordon equation: in fact in the continuous limit, with a suitable change of the variables, the equation can be put in the form $\phi_{tt} - \phi_{xx} + \sin(\phi) = 0$ and in turn this equation, with the changes $2\alpha = x + t$ and $2\beta = x - t$ takes the usual form $\phi_{\alpha\beta} = \sin(\phi)$. More than ten years after the publication of Frenkel and Kontorova results, Alfred Seeger, while working on his Phd thesis, became aware by chance of the works of Bianchi on sine Gordon equation. So a number of well known solitonic features, such as the preservation of shape and velocity after collisions, were obtained [104] by means of the permutability theorem. In 1967 Lamb [67] derived the sine Gordon equation as a model for optical pulse propagation in a two energy level medium having relaxation times which are long compared to pulse length (ultrashort optical pulses). Lamb was aware of the Seeger works, so in 1971 [68] he used the permutabilty theorem to analyse the decomposition, experimentally observed, of "$2N\pi$" pulses into $N$ stable "$2\pi$" pulses. The situation became even more interesting after the work of Wahlquist and Estabrook [122] on Bäcklund transformations for the KdV equation of the 1973. In fact not only they found the transformations and the associated permutability theorem for the KdV equation, but moreover they stressed how an iteration of this theorem can analytically describe the behavior of the soliton solutions numerically observed by Zabusky and Kruskal in 1965 [131]. Furthermore a connection with the incoming Inverse Spectral Transform theory was established. Let me summarize their findings.

They rewrote the KdV equation:

$$u_t + (6u^2 + u_{xx})_x = 0$$

by introducing the potential function defined by $u = -w_x$. This potential function satisfies the equation:

$$w_t = 6w_x^2 - w_{xxx},$$

Given a solution $u$ of the KdV and then the associated potential $w$, another solution $u_1$ with potential $w_1$ can be found by the following Bäcklund transformations:

$$\begin{aligned} (w_1 + w)_x &= (w_1 - w)^2 - \lambda_1 \\ (w_1 + w)_t &= 4\left(\lambda_1 u_1 + u^2 - u(w_1 - w)^2 - u_x(w_1 - w)\right) \end{aligned} \tag{1.36}$$

where $\lambda_1$ is an arbitrary parameter. The permutability theorem allowed them to find a relation between the elements of a *soliton ladder*. In particular by considering subsequent transformations induced by (1.36) with different parameters, for example the transformation from $u$ to $u_1$ with $\lambda_1$ and then from $u_1$ to $u_{12}$ with parameter $\lambda_2$, they expressed the $n$th step of the ladder by the recursion relation:

$$w_n = w_{n-2} + \frac{\lambda_n - \lambda_{n-1}}{w_{(n-1)'} - w_{n-1}} \tag{1.37}$$

where the subscript $n$ denotes the set of $n$ parameters $\{\lambda_1, \dots \lambda_n\}$ and $n'$ the set $\{\lambda_1, \dots \lambda_{n-1}, \lambda_{n+1}\}$ (with $w_0 = w$). So for $n = 3$ one has:

$$w_{123} = w + \frac{\lambda_3^2 - \lambda_2^2}{w_{13} - w_{12}},$$

that can be obviously expressed in terms of only single soliton solutions by an iteration of the formula (1.37) in the case $n = 2$. Note that the first of the equations (1.36) has the form of a Riccati equation: in fact, by posing $v = w_1 - w$, one has:

$$v_x + 2u = v^2 - \lambda_1.$$

The linearization of this equation by the substitution $v = -\frac{\psi_x}{\psi}$ gives:

$$\psi_{xx} + (2u - \lambda)\psi = 0,$$

that is the Schrödinger equation: this result gave the connection between the Bäcklund transformation of the KdV and the outstanding observation by Gardner, Green, Kruskal and Miura [42] that the solutions of the KdV equation itself are related with the potential of the Schrödinger equation. In 1974, one years after the work of Wahlquist and Estabrook, Lamb [69], by applying the Clairin method, found the Bäcklund transformations for the NLS equation. Again a permutability theorem was obtained (although, in the words of Lamb [69] "*the result appears to be too complex to be useful for computational purposes*") and a connection with the linear equations for the inverse problem associated with the NLS equation was established. From now on a lot of results on Bäcklund transformations for many classes of integrable partial differential equations were obtained; for a review see [101]. At this point it was clear that there are some characteristic properties common to all integrable equations: they possess a Lax representation, which we will analyze later, are solvable by inverse scattering transform and possess Bäcklund transformations. Nevertheless, as noted first by Wojciechowski in 1982 [126], although many *finite* dimensional systems also admit Lax representation and are completely integrable, the analogue of Bäcklund transformations for these systems was not known. So in his aforementioned work he provided the Bäcklund transformations for the classical Calogero-Moser system. There he clearly noticed how the Bäcklund transformations for finite dimensional system can be seen as canonical transformations preserving the algebraic form of the Hamiltonian. Really this is not an accident as it will be clarified in 1.5.

## 1.4 Bäcklund transformations and the Lax formalism

The *Renaissance* of Bäcklund transformations matches with the *golden age* of the integrability theory and of the associated inverse spectral methods. As it is well known, in the later sixties fundamental developments were obtained in the theory of nonlinear differential equations. On the one hand in [42] were derived explicit solutions of the KdV equation and was described the interaction of an arbitrary number of solitons, on the other hand Peter Lax in [71] introduced an operatorial compatibility condition that subsequently allowed to extend the method adopted in [42] to solve a number of nonlinear evolution equations with ubiquitous physical applications. Let me summarize the mean features of the Lax method in order to well understand the connections with the Bäcklund transformations theory. The prototypical example of Lax pair is the one that generates the KdV equation. One introduces two linear problems associated to two operators $L$ and $M$ as follows:

$$L\phi = \lambda\phi, \qquad L = \partial_x^2 + u(x,t), \tag{1.38a}$$

$$\phi_t = M\phi, \qquad M = \gamma - 3u_x - 6u\partial_x - 4\partial_{xxx}. \tag{1.38b}$$

Here $\lambda$ is the spectral parameter and $\gamma$ is an arbitrary constant. The function $\phi$, that according to (1.38a) can be reads as a wave function for the Schrödinger equation with potential $u(x,t)$, depends on $x$, $t$ and $\lambda$. The compatibility equations for the wave function $\phi$ lead to the Lax equation:

$$L_t + LM - ML = L_t + [L, M] = 0 \tag{1.39}$$

and this in turns is equivalent to the KdV equation. The core of the inverse spectral method, the spectral analysis, is derived from the study of equation (1.38a). The physical interpretation of the method is well described by Fokas; by using his words [38]:"*Let KdV describe the propagation of a water wave and suppose that this wave is frozen at a given instant of time. By bombarding this water wave with quantum particles, one can reconstruct its shape from knowledge of how these particle scatter. In other words, the scattering data provide an alternative description of the wave at a fixed time*". Once the scattering data have been found, it is possible to compute their time dependence thanks to (1.38b), and so insert the time dependence in the solution of the KdV. More precisely, given $u(x,0)$, the spectrum of the Schrödinger equation (1.38a) is given by a finite number of discrete eigenvalues, say $\lambda = \{\kappa_n^2\}_{n=1}^N$ for $\lambda > 0$ and a continuum set, $\lambda = -k^2$, for $\lambda < 0$. The asymptotics of the corresponding eigenvectors (at $t = 0$) can be written as follows:

$$\begin{array}{llll}
\lambda > 0; & x \to -\infty & \phi_n(x,0,\kappa_n) \sim c_n(0)e^{-\kappa_n x} \quad \text{with} & \displaystyle\int_{-\infty}^{+\infty} \phi_n^2(x,0,\kappa_n)dx = 1; \\
\lambda < 0; & x \to -\infty & \phi(x,0,k) \sim T(k,0)e^{-ikx}, & \\
 & x \to +\infty & \phi(x,0,k) \sim e^{-ikx} + R(k,t)e^{ikx}, &
\end{array}$$

where $T(k,t)$ and $R(k,t)$ are the transmission and reflection function for the wave function $\phi$. The time evolution of these functions and of $c_n(t)$ can be found by equation (1.38b); the result is $c_n(t) = c_n(0)e^{4\kappa_n^3 t}$, $T(k,t) = T(k,0)$ and $R(k,t) = R(k,0)e^{8ik^3 t}$. At this point the scattering data are completely described by the set:

$$S(\lambda, t) = (\kappa_n, c_n(t), R(k,t)) .$$

The link between the corresponding solution of the KdV and this data set is given by a *linear* integral equation; indeed if one defines the function $F(x,t)$ by:

$$F(x,t) = \sum_{n=1}^{N} c_n^2(t) e^{-\kappa_n x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} R(k,t) e^{ikx} dk,$$

then it solves the Gel'fand-Levitan-Marchenko equation:

$$K(x,y,t) + F(x+y,t) + \int_x^{\infty} K(x,s,t) F(s+y,t) ds = 0$$

and the function $u(x,t)$ is reconstructed by:

$$u(x,t) = 2\frac{\partial}{\partial x} K(x,x,t).$$

Now the connection with Bäcklund transformations. Suppose to have two different solutions, $u$ and $\tilde{u}$, to the KdV equation. Correspondingly to these solutions there must exist two different spectral problems, the first given by the equations (1.38a) and (1.38b), and the other by:

$$\tilde{L}\tilde{\phi} = \lambda\tilde{\phi}, \qquad \tilde{L} = \partial_x^2 + \tilde{u}(x,t), \tag{1.40a}$$

$$\tilde{\phi}_t = \tilde{M}\tilde{\phi}, \qquad \tilde{M} = \gamma - 3\tilde{u}_x - 6\tilde{u}\partial_x - 4\partial_{xxx}. \tag{1.40b}$$

Suppose also that $u$ and $\tilde{u}$ are linked by a Bäcklund transformation. The relation between the two solutions defined by this transformation reflects into a relation between the wave functions of the two spectral problems. This means that it has to exist an operator $D$, that we will call the *dressing operator* or *dressing matrix* and that depends on $u$, $\tilde{u}$ and $\lambda$, such that

$$\tilde{\phi} = D\phi. \tag{1.41}$$

Inserting this equation in (1.40a) and taking into account (1.38a), one obtains the equation for the Bäcklund transformations in the Lax formalism:

$$\boxed{\tilde{L}D = DL} \tag{1.42}$$

As we will see, this boxed equation will be of fundamental importance for the core of this work. Obviously, given a dressing operator $D$ fulfilling (1.42), one has to ensure also that (1.40b) is fulfilled. For differential equations possessing a Lax representation, the problem of finding Bäcklund transformations reduces to the problem of finding the corresponding dressing operator.

## 1.5 Bäcklund transformations and integrable discretizations

### 1.5.1 Integrable discretizations

Cellular automata, neural networks and self-organizing phenomena are only few of the key notions appearing in the modern developments of discrete dynamics. One of the main practical interest in integrable discretizations of nonlinear evolution equations arises from the needs of computational physics. The problem is to construct a discrete analogue of the continuum model preserving its mean features. In statistical mechanics, for obvious reasons, it is of fundamental importance that the long-term dynamics of the continuous model can be related to the corresponding dynamics of the discrete system. An encyclopedic work on the Hamiltonian approach to integrable discretization is that of Suris [115]. As a matter of fact the problem of integrable discretizations is not to solve the discrete dynamics, but rather to find what is the most appropriate discrete counterpart of a continuous system. Since in this work we will deal only with integrable system, for the sake of completeness I will first recall the Liouville-Arnold theorem on complete integrable systems and then I will specify, following Suris, what is meant by "appropriate" discretization.

**Theorem 1** *Suppose to have an autonomous Hamiltonian system (with Hamiltonian $H$) with $n$ degree of freedom (the dimension of phase space is then $2n$) and with $n$ independent first integrals in involution, that is $n$ functions $I_k$, $k = 1 \dots n$, such that the gradients $\nabla I_k$ are $n$ independent vectors for every point of the phase space and the Poisson bracket $\{I_k, I_m\}$ vanishes for every $k, m = 1 \dots n$. Consider the level set*

$$M_a = \{x \in \mathbb{R}^{2n} : I_k = a_k, k = 1 \dots n\},$$

*where $a \in \mathbb{R}^n$. Then:*

- *$M_a$ is a smooth manifold invariant under the phase flow associated with $I_1, \dots I_n$;*
- *If $M_a$ is compact and connected it is diffeomorphic to an $n$-dimensional torus, that is the set $T^n$ of $n$ angular coordinates:*

$$T^n = \{\phi_1, \dots \phi_n\};$$

- *The flow with respect to $H$ determines a quasi-periodic motion on $M_a$:*

$$\frac{d\phi_i}{dt} = \omega_i, \qquad \omega_i = \omega_i(I_j);$$

- *The equations of motion with respect the Hamiltonian $H$ can be integrated by quadratures.*

For the detailed proof of these statements the reader can see for example [8].
Now, having in mind the precise formulation of Suris [115], we can state the problem of integrable discretization as in the following. Suppose to have an autonomous complete integrable system, governed by an Hamiltonian $H$, and denote simply by $x$ the dynamic variables of this system. Let $I_k(x)$ be the integrals in involution. The equations of motion will be given by:

$$\dot{x} = \{H, x\} = f(x). \tag{1.43}$$

The "appropriate" discrete counterpart of this system will be a family of maps:

$$\tilde{x} = \Phi(x, \mu)$$

depending smoothly on a parameter $\mu$ and such that:

- In the limit $\mu \to 0$ the map approximate the flow (1.43):
$$\Phi(x, \mu) = x + \mu f(x) + O(\mu^2).$$
- The map is Poisson with respect to the bracket $\{\cdot, \cdot\}$ or some its deformation $\{\cdot, \cdot\}_\mu = \{\cdot, \cdot\} + O(\mu)$.
- The map is integrable and the integrals approximate those of the continuous system: $I_k(x, \mu) = I_k(x) + O(\mu)$.

Note that it is not requested the explicitness of the map, nor the conservation of the orbits.
If the more restrictive conditions $\{\cdot, \cdot\}_\mu = \{\cdot, \cdot\}$ and $I_k(x, \mu) = I_k(x)$ are fulfilled, than I will talk about *exact-time discretization*: as will be showed in the chapter (4), at least in some special cases of such discretization for the Kirchhoff top, and as a conjecture for the exact time discretization of the Kirchhoff top as a whole, it will be possible to preserve also the physical orbits of the system.
A number of methods have been proposed in the course of time to establish a modus operandi in discretizing continuous flows. A complete list can be found in [115] (see also [116]); some of these approaches are:

- The Ablowitz-Ladik approach [1], [2]: if an integrable system can be written as the compatibility condition for two associated linear problem, then the corresponding discrete system can be found by discretizing, in some way, one or both of them. "In some way" indeed means that this can be done in various ways;

Faddeev and Takhtajan [35] try to get some fixed rule by focusing on Hamiltonian properties of the model considered: a common feature for models in $1+1$ dimensions is to retain the $r$-matrix and substitute the linear Poisson bracket with the quadratic one (see appendix B for a discussion on linear and quadratic $r$-matrix structures).

- The Hirota method: it is based on the bilinear approach introduced by Hirota [51] and widely used to obtain soliton solutions of non linear evolution equations. It seems to have some connections with a method proposed by Kahan but that has remained largely ignored [87]. As noted in [115], the mechanism behind the method is yet to be fully understood.

- The Moser and Veselov approach [78], [129], [127]: it is based on discrete lagrangian equations obtained by means of variational principles. It is also known as the factorization method and is indeed based on some observations of Symes [119] on the connection of Toda flow with the $QR$-algorithm, an important tool in the numerical analysis for the diagonalization of matrices. Moser and Veselov works gave rise to a widespread and renewed interest in the theory of integrable maps within the mathematical physics community.

- Geometric method [118], [23], [26]: as we have seen in the first part of this work, there is a deep connection between geometry of surfaces and integrable differential equations. It is natural then to check what is obtained by discretizing the notions and methods of smooth surface theory. In my opinion this could be one of the more fruitful direction of the future research.

- The Bäcklund transformations method: the approach that represents the object of the thesis and that will be extensively explained in the next paragraph. In my perspective this is the most satisfactory and efficient method to obtain discrete version of integrable non linear evolution equations that admit a Lax representation; the deep connection with the geometry of surfaces should not be underestimated as a source of new discoveries and new queries.

### 1.5.2 The approach à la Bäcklund

Quite remarkably, Bäcklund transformations provide a powerful tool in the discretization of integrable differential equations. The idea behind this technique is very simple: suppose that a differential equation possesses an associated Lax structure and a Bäcklund transformation. By viewing the new solution as the old one but computed at the next time-step, then the Bäcklund transformation becomes a differential-difference (or only difference) equation. The same argument can be repeated also at the level of the Lax matrices, so that one is often able to obtain also the Lax pair for the discrete system, showing in this way its integrability. To the best of my knowledge, one of the first clear evidences of the capability of this point of view was given by Levi and Benguria in [72], where these lines of reasoning were used to show that the following differential difference approximation of the KdV equation:

$$(w(n+1,t)+w(n,t))_t + [w(n+1,t)-w(n,t)]\left[h+\frac{1}{2}(w(n+1,t)-w(n,t))\right]$$

is indeed integrable. However, in my opinion, in the course of all the eighties the full potential of the Bäcklund transformations was not well understood. As yet mentioned in 1.3, in 1982 Wojciechowski [126] sought to find the analogues of the Bäcklund transformations for finite dimensional systems. As a matter of fact he found the Bäcklund transformations for a well known many body system, the so called Calogero-Moser system [27], [77]. His results are noteworthy, but they were forgotten for some time. Furthermore I'm quite sure that Wojciechowski wasn't aware of Levi and Benguria's work: in fact only in 1996 it was realized by Nijhoff, Ragnisco and Kuznetsov [84] that indeed the discrete Calogero-Moser model can be inferred from the Bäcklund transformations given in [126]. Let me summarize for completeness the results of Wojciechowski. He considered a set of systems of $N$ interacting particles on a line with the following two-body potentials:

$$\begin{cases} \text{a)} \quad V(x) = \wp(x), \\ \text{b)} \quad V(x) = \dfrac{1}{x^2}, \left(\coth^2(x), \cot^2(x)\right), \\ \text{c)} \quad V(x) = \dfrac{1}{x^2} + w^2x^2. \end{cases} \tag{1.44}$$

where $\wp(x)$ is the Weierstraß elliptic function. The Bäcklund transformations for these systems are given by the following expressions:

$$\begin{aligned} \dot{x}_k &= 2\sum_{j\neq k}^{M}\psi(x_k-x_j) - 2\sum_{j=1}^{N}\psi(x_k-y_j) + 2\lambda - wx_k \\ \dot{y}_m &= 2\sum_{j=1}^{M}\psi(y_m-x_j) - 2\sum_{j\neq m}^{N}\psi(y_m-y_j) + 2\lambda - wy_m \end{aligned} \tag{1.45}$$

where $k = 1 \dots M$, $m = 1 \dots N$ and correspondingly to the case a), b) and c), the function $\psi$ takes the following values:

$$\begin{cases} \text{a)} \quad \psi(x) = \zeta(x), M = N, w = 0, \\ \text{b)} \quad \psi(x) = \dfrac{1}{x}, (\coth(x), \cot(x)), M, N \,\text{arbitrary}, w = 0, \\ \text{c)} \quad \psi(x) = \dfrac{1}{x}, M, N \,\text{arbitrary}, w \neq 0. \end{cases} \tag{1.46}$$

Here $\zeta(x)$ is the Weierstraß zeta function. Wojciechowski was able to show that indeed the conditions of compatibility reduce to the dynamical equations and that the transformations provide an algebraic construction of new solutions thanks to a permutabilty theorem. Furthermore the transformations (1.45) are canonical because there exist a generating function $F$ such that $\dot{x}_k = \frac{\partial F}{\partial x_k}$ and $\dot{y}_k = -\frac{\partial F}{\partial y_k}$. This canonical transformation has also the property to preserve the algebraic form of the Hamiltonian (obviously other than the Hamiltonian character of the equations of motion). For a twist of fate the work of Wojciechowski was recognized in 1996 to discretize the corresponding continuous flow in [84], a work which proposed the discretization of a relativistic variant of the Calogero-Moser model, the so called Ruijsenaars-Schneider model, with transformations that only after, in [64], was recognized to be indeed the Bäcklund transformations for such model. As for the discretization of partial differential equations by means of Bäcklund transformations, the situation was quite similar: in fact some results appear in 1982 [81] and 1983 [83] but only later the relevance of Bäcklund transformations and permutabilty theorems in discretizing PDEs were fully acknowledged [26][82]. So in 1983 Nijhoff, Quispel and Capel [83] found a difference-difference version of some nonlinear PDEs of physical interest in 1+1 dimension. Among them there was the KdV equation. Unlike Wojciechowski now the authors were aware of the work of Levi and Benguria so that they could establish quite clearly (although, it seems, by following an independent way) the connection of their discretization with the Bäcklund transformations and the Bianchi permutability theorem. So if $u_{n,m}$ represents the dynamical variable at site $(n, m)$, where $n, m \in \mathbb{Z}$, the lattice version of the KdV reads (see also [82]):

$$(p - q + u_{n,m+1} - u_{n+1,m})\,(p + q - u_{n+1,m+1} + u_{n,m}) = p^2 - q^2 \qquad (1.47)$$

where $p, q \in \mathbb{C}$ are two parameters. As shown in [83], this is equivalent to the Bianchi permutability theorem: by combining two Bäcklund transformations for the KdV, as given in [81], the first, say $\tilde{u}$, with parameter $p$ and the second one, say $\hat{u}$, with parameter $q$, then one obtains the formula:

$$\left(p + q - \hat{\tilde{u}} + u\right)(p - q + \hat{u} - \tilde{u}) = p^2 - q^2$$

which is equivalent to the lattice KdV equation (1.47): so as stated clearly in [82], the iteration of Bäcklund transformations leads to a lattice of transformed fields $u$ and the Bianchi permutabilty theorem is nothing but consistency condition on the lattice for the *partial difference equation*.

In the course of 1990s, in the wake of Veselov works on *Lagrange correspondences* [127], [128], [129], that is multi-valued symplectic maps that have enough integrals of motion and that are time discretizations of some known classical Liouville integrable systems, a lot of results on discretization of finite dimensional integrable systems were achieved: for the Ruijsenaars-Schneider model [84], the Henon-Heiles, Garnier and Neumann systems [91], [92], [53], the Euler top [24], the Lagrange top [25], the rational Gaudin magnet [52] and others (see also the excellent book by Suris [115] and the references therein). It turned out [65] that almost all the discretizations for these systems associate new solutions to a given one: it was clear then that they are Bäcklund transformations for such systems. These developments suggested to Sklyanin and Kuznetsov that the concept of Bäcklund transformations could be revised in order to highlight some new aspects. Actually in the previous times these two authors were very active in the field of separation of variables and its connection with the new techniques of classical and quantum inverse scattering method, so not only they were able to clearly elucidate the role of Bäcklund transformations in the context of finite dimensional integrable systems, but they also established deep and fruitful connections with Hamiltonian dynamics and separation of variables. I think that all the potentialities of these new ideas are not yet fully exploited, and I hope to give in this work some new light on the geometric, mechanical and Hamiltonian meaning of Bäcklund transformations for finite dimensional systems. The fundamental paper that now I will survey is [64] (see also [65]). For a more detailed account on the links between the inverse scattering method and the classical notion of separation of variables the reader can see [109].

Suppose to have a classical integrable dynamical system described by a Lax matrix $L(\lambda)$, where $\lambda$ is the spectral parameter, and that the commuting Hamiltonians $H_i$ can be obtained by the coefficients of the characteristic polynomial $\det(L(\lambda) - \gamma \mathbb{1})$. Let the dynamical variables be $q_i$ and $p_i$, $i = 1 \ldots N$, and assume for simplicity that these variables are canonical:

$$\{q_i, q_j\} = 0, \qquad \{p_i, p_j\} = 0, \qquad \{q_i, p_j\} = \delta_{ij}.$$

Correspondingly to the matrix $L(\lambda)$ it has to exist the Lax matrix $\tilde{L}(\lambda)$ of the transformed variables, that is $\tilde{L}(\lambda) \doteq L(\lambda, \tilde{q}, \tilde{p})$. As repeatedly noticed [37] [126], [102], [127], [65], [84], [78], [75] [4] [110], [111], [36] the Bäcklund transformations can be seen as canonical transformations preserving the algebraic form of the Hamiltonians. Since the characteristic polynomial is the generating function of the integrals of motion, their invariance amounts to require the existence of a similarity matrix, the *dressing matrix*, that intertwines the two Lax matrices (see also (1.42)):

$$\tilde{L}(\lambda)D(\lambda) = D(\lambda)L(\lambda). \tag{1.48}$$

Obviously the matrix $D(\lambda)$ needs not to be unique because a dynamical system can have different Bäcklund transformations. More remarkably it is possible to have one-parametric or multi-parametric families of Bäcklund transformations: as we will see this is related to the existence of a dressing matrix $D(\lambda)$ such that $\det(D(\mu)) = 0$, where $\mu$ is a particular (but arbitrary) value of the spectral parameter $\lambda$; it is not an overstatement to say that this fact is at the core of the Sklyanin and Kuznetsov speculations: indeed it allows them also to introduce a new property of Bäcklund transformations, that is the *spectrality*. But let me go with order.

Let me assume to be in the simplest nontrivial case, namely the case of $2 \times 2$ matrices. Assume also that it is possible to find a parametrization of the matrix $D(\lambda)$ such that its determinant is zero when $\lambda = \mu$. This means that $D(\mu)$ is a rank one matrix. Let $|\Omega(\mu)\rangle$ be the corresponding kernel. By acting with it on the equation (1.48), one finds:

$$\tilde{L}(\mu)D(\mu)|\Omega(\mu)\rangle = 0 \Rightarrow D(\mu)\left(L(\mu)|\Omega(\mu)\rangle\right) = 0.$$

But this implies that the vector $L(\mu)|\Omega(\mu)\rangle$ is proportional to the kernel $|\Omega(\mu)\rangle$, that is:

$$L(\mu)|\Omega(\mu)\rangle = \gamma(\mu)|\Omega(\mu)\rangle$$

and, in turns, this means just that $|\Omega(\mu)\rangle$ is also the eigenvector, often called the Baker-Akhiezer function, of $L(\mu)$ with eigenvalue $\gamma(\mu)$, so that the characteristic polynomial evaluated at $\lambda = \mu$ is zero:

$$\det(L(\mu) - \gamma \mathbb{1}) = 0. \tag{1.49}$$

This seems to be an harmless equivalence but actually it is the *separation equation*, in the sense of Hamilton Jacobi separability, for the dynamical system. Let me explain this point. The eigenvalue $|\Omega(\mu)\rangle$ is defined up to a multiplicative factor, so it is possible to define a normalization for it. Fix this normalization by introducing the vector $|\alpha(\mu)\rangle$:

$$\langle\alpha(\mu)|\Omega(\mu)\rangle = 1.$$

In general the vector $|\alpha(\mu)\rangle$ can depend also on the dynamical variables. With the above normalization $|\Omega(\mu)\rangle$ becomes a meromorphic function on the surface defined by $\det(L(\mu) - \gamma(\mu)\mathbb{1}) = 0$ (obviously to any fixed $\mu$ there correspond 2 possible values of $\gamma(\mu)$ for a $2 \times 2$ Lax matrix). At this point there is the crucial observation: the poles of the eigenvector $|\Omega(\mu)\rangle$, say at $\mu = x_j$ can be explicitated and Poisson commute, the corresponding eigenvalue of $L(x_j)$, say $\gamma_j = \gamma(x_j)$, or in general functions of $\gamma_j$, together with the variables $x_j$, are a set of *separated* canonical variables for the dynamical system described by the Lax matrix $L(\lambda)$. This means that:

**a)** the Poisson brackets for $x_j$ and $\gamma_j$ are canonical:

$$\{x_i, x_j\} = 0, \qquad \{\gamma_i, \gamma_j\} = 0, \qquad \{x_j, \gamma_i\} = \delta_{ij}. \tag{1.50}$$

**b)** there exist a set of $N$ relations binding together each pair $(x_j, \gamma_j)$ with the Hamiltonians of the system $H_i$, $i = 1 \ldots N$.

In order to prove the first assertion one needs of the Poisson brackets between the entries of $L(\lambda)$, and these are usually provided by the $r$-matrix. For example let me suppose to have the $2 \times 2$ Lax matrix:

$$L(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}$$

satisfying the following Poisson brackets (for details on $r$-matrix formalism see appendix B):

$$\{L(\lambda) \otimes \mathbb{1}, \mathbb{1} \otimes L(\mu)\} = [r(\lambda - \mu), L(\lambda) \otimes \mathbb{1} + \mathbb{1} \otimes L(\mu)] \tag{1.51}$$

where, by definition, $\{L(\lambda) \otimes \mathbb{1}, \mathbb{1} \otimes L(\mu)\}$ is given by:

$$\{L(\lambda) \otimes \mathbb{1}, \mathbb{1} \otimes L(\mu)\}_{jk,mn} = \{L(\lambda)_{jm}, L(\mu)_{kn}\}$$

and $r(\lambda)$ is the classical rational $r$-matrix, proportional to the permutation operator $P$ in $\mathbb{C}^2 \otimes \mathbb{C}^2$, defined by [35], [9]:

$$r(\lambda) = \frac{P}{\lambda}, \qquad P(\phi \otimes \psi) = \psi \otimes \phi, \qquad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

These Lax and Poisson structures are actually common to a number of integrable systems (see for example[35], [9]) including the rational Gaudin model [52]. The explicit Poisson brackets for the matrix elements $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$, corresponding to (1.51), can be easily computed as:

$$\{A(\lambda), A(\mu)\} = \{B(\lambda), B(\mu)\} = \{C(\lambda), C(\mu)\} = 0, \tag{1.52}$$

$$\{A(\lambda), B(\mu)\} = \frac{B(\mu) - B(\lambda)}{\lambda - \mu}, \tag{1.53}$$

$$\{A(\lambda), C(\mu)\} = \frac{C(\lambda) - C(\mu)}{\lambda - \mu}, \tag{1.54}$$

$$\{B(\lambda), C(\mu)\} = \frac{2(A(\mu) - A(\lambda))}{\lambda - \mu}. \tag{1.55}$$

It turns out [109] that in this case any constant numeric vector of normalization $|\alpha(\mu)\rangle$ is able to produce a new set of separated variables. For example if one takes $\langle\alpha| = (1, 0)$, then the equation for the poles of the Baker-Akhiezer function reads:

$$\langle\alpha(x_j)|\Omega(x_j)\rangle = 0.$$

It is straightforward to see that the compatibility of this equation with the corresponding eigenvector relation $L(x_j)|\Omega(x_j)\rangle = \gamma(x_j)|\Omega(x_j)\rangle$ gives:

$$B(x_j) = 0 \qquad \gamma(x_j) = -A(x_j)$$

Now by solving $B(x_j)$ with respect to $x_j$, one obtains the first set of *commuting* variables. Indeed they commute as a consequence of the second relation in(1.52), $\{B(\lambda), B(\mu)\} = 0$, which readily implies the commutativity of the $B(\mu)$'s zeroes:

$\{x_i, x_j\} = 0$. Equivalently, thanks to the first relation in (1.52), one has the Poisson commutativity of the variables $\gamma_j = -A(x_j)$. For the commutation relations between $x_i$ and $A(x_j)$ one has to use (1.53). First of all let me find the Poisson bracket between $x_j$ and any function $f$. By using $B(x_j) = 0$, one clearly has:

$$\{f, B(x_j)\} = 0 = \{f, B(\mu)\}|_{\mu=x_j} + B'(x_j)\{f, x_j\} \quad \Rightarrow \{f, x_j\} = -\frac{\{f, B(\mu)\}|_{\mu=x_j}}{B'(x_j)}$$

where in the calculation of the bracket $\{f, B(\mu)\}$ one considers $\mu$ as a constant. Then for $\{A(x_i), x_j\}$ one has:

$$\{A(x_i), x_j\} = \{A(\lambda), x_j\}|_{\lambda=x_i} + A'(x_i)\{x_i, x_j\} = \{A(\lambda), x_j\}|_{\lambda=x_i}$$

because $\{x_i, x_j\} = 0$. By putting all together:

$$\{A(x_i), x_j\} = -\frac{\{A(\lambda), B(\mu)\}|_{\substack{\lambda=x_i\\ \mu=x_j}}}{B'(x_j)} \overset{1.53}{=} \frac{1}{B'(x_j)} \frac{B(x_i) - B(x_j)}{(x_i - x_j)}.$$

This relation clearly vanishes for $x_i \neq x_j$ because $B(x_i) = B(x_j) = 0$. For $i = j$, by l'Hôpital's rule, one readily obtains $\{A(x_i), x_j\} = \delta_{ij}$, in agreement with (1.50).
The second assertion, that is the existence of a set of $N$ relations binding together each pair $(x_j, \gamma_j)$ with the Hamiltonians of the system, is self-evident: since $\gamma(x_j)$ is an eigenvalue of $L(x_j)$, the pair $(x_j, \gamma_j)$ satisfies $\det(L(x_j) - \gamma(x_j)\mathbb{1}) = 0$. This is just a relation between the canonical coordinates $(x_j, \gamma_j)$ and the $N$ Hamiltonians $H_j$ (recall that by hypothesis the Hamiltonians are the coefficients of the characteristic equation): it can be used in the usual Hamilton-Jacobi method in order to separate the variables [8]. If one has $N$ pairs $(x_j, \gamma_j)$, then the separation is complete (in the previous example this is the case if $B(\mu) = 0$ has $N$ distinct roots). These observations clarify why (1.49) can be seen as the separation equation of the dynamical system.
The strength and attractiveness of all this construction is that it has an almost clear quantum counterpart [112], [64] [108]. The involutivity of the integrals of motion is replaced by the commutativity of the corresponding quantum operators:

$$\{H_i, H_j\} = 0 \quad \Longrightarrow \quad [H_i, H_j] = 0 \qquad i, j = 1 \dots N.$$

The set of classical separated coordinates, $\{x_i, \gamma_i\}_{i=1}^N$ satisfies the set of $N$ separation equations $\det(L(x_i) - \gamma(x_i)\mathbb{1}) = 0$. Here a problem of ordering operators arises. In the separation equations the canonical coordinates and the integrals appear; emphasizing these dependencies, the relations $\det(L(x_i) - \gamma(x_i)\mathbb{1}) = 0$ can be written in general as:

$$\Phi_i(x_i, \gamma_i, H_1, \dots H_N) = 0, \qquad i = 1 \dots N. \tag{1.56}$$

The commutators between the $x_j$'s and the $\gamma_j$'s are obviously given by:

$$[x_i, x_j] = 0, \quad [\gamma_i, \gamma_j] = 0, \quad [\gamma_i, x_j] = -i\hbar\delta_{ij}.$$

Suppose that the ordering in (1.56) is exactly as it appears. Since the operators $H_i$ commute, they will have a common eigenfunction, say $\Psi$. Applying these eigenfunction to (1.56), one readily realizes that $\Psi$ satisfies the set of $N$ equations:

$$\Phi_i(x_i, -i\hbar\frac{\partial}{\partial x_i}, h_1, \dots h_N)\Psi = 0, \tag{1.57}$$

where $h_i$, $i = 1 \dots N$ are the eigenvalues corresponding to $H_i$. The eigenfunction $\Psi$ is then factorized into functions of one variables:

$$\Psi = \prod_{j=1}^{N} \psi_i(x_i)$$

each satisfying the differential equation:

$$\Phi_i(x_i, -i\hbar\frac{\partial}{\partial x_i}, h_1, \dots h_N)\psi_i(x_i) = 0.$$

It is possible to establish also a connection between Bäcklund transformations and the so-called Baxter's $Q$-operator [85], [112]; this operator, that is indeed a family of operators depending on a parameter, say $\mu$, commutes with the Hamiltonians of the system and its eigenvalue solves the *Baxter's equation*, that can be considered the quantum counterpart of the separation equation (1.49). Indeed the Baxter equation allows, exactly as in the classical separation of variables, to determine the spectrum of commuting Hamiltonians, reducing the spectral problem to a set of one-dimensional problems. Experience shows [112] that the classical Bäcklund transformations are exactly the similarity transformations induced by the Baxter's operator. In fact, since the Bäcklund transformations are canonical maps, it is possible to describe them by a generating function: if $\{q_i, p_i\}$ are the dynamical variables for our system and $\{\tilde{q}_i, \tilde{p}_i\}$ are the Bäcklund transformed variables, then must exist a generating function, say $F_\mu(q, \tilde{q})$ such that:

$$p_i = \frac{\partial F_\mu}{\partial q_i}, \qquad \tilde{p}_i = -\frac{\partial F_\mu}{\partial \tilde{q}_i}$$

the subscript $\mu$ on $F$ denotes the possibility that $F$ can indeed depends on a parameter $\mu$ because, as explained before, in general one has a parametric family of Bäcklund transformations. The quantum analog of these canonical transformations is provided by a similarity transformations:

$$p_i = Q_\mu q_i Q_\mu^{-1}, \qquad \tilde{p}_i = Q_\mu \tilde{q}_i Q_\mu^{-1}.$$

where $Q_\mu$ is an integral operator whose kernel, say $K_\mu$, corresponds to the generating function $F_\mu$ through the semiclassical relation:

$$K_\mu \sim e^{-\left(\frac{i}{\hbar}F_\mu\right)}.$$

In my opinion all these evidences of the connections between the Bäcklund transformations and Baxter's $Q$ operator need of a profound theoretical setting in order to exploit all the possible implications that can arise from a deeper understanding.
On the account of this brief synopsis on the separation of variables, the means of the *spectrality property* introduced by Kuznetsov and Sklyanin [64] can be better understood. In few words, this property amounts to the following observation [64]: if $\gamma$ is the variable conjugated to $\mu$:

$$\gamma = -\frac{\partial F_\mu}{\partial \mu}, \tag{1.58}$$

then for some function $g(\gamma)$ one has $\det(L(\mu) - g(\gamma)\mathbb{1}) = 0$. Notice that this last equation is just the separation equation (1.49): the canonical conjugation between the variables $\mu_j$'s and $\gamma_j$'s gives an Hamiltonian interpretation of the relation (1.58). In the general statement one needs of a function $g$ because, roughly speaking, it is possible that the variables conjugated to the $\mu_j$'s are some function of the $\gamma_j$'s. As we will see in the discussion of the discretization of the Gaudin and Kirchhoff models, it is possible to give also another mechanical interpretation of the equation (1.58). Suppose in fact that the Bäcklund transformation is connected to the identity by the parameter $\mu$, so we assume that the map is a smooth function of $\mu$ and there exist a value of this parameter, say for simplicity $\mu = 0$, such that

$$\lim_{\mu \to 0} h(\tilde{q}, \tilde{p}) = h(q, p)$$

for any function $h$ of the dynamical variables. One can now consider the relation defining the map, $\tilde{L}D = DL$ around the point $\mu = 0$; the Taylor series for the matrix $D$ will be $D = \mathbb{1} + \mu D_0 + O(\mu^2)$, because it is connected to the identity; by defining

$$\dot{L} \doteq \lim_{\mu \to 0} \frac{\tilde{L} - L}{\mu},$$

one readily obtains:

$$\dot{L} = [D_0, L],$$

that is the Lax equation for a continuous flow. Note that $D_0$ can depend on a parameter, so that one has a family of flows. For the Gaudin models and the Kirchhoff top it turns out that this flows are indeed equivalent just to those governed by the Hamiltonian function $\gamma$: in this sense, if $\mu$ is the evolution parameter (time), the relation between $\gamma$ and $\mu$ is just $\gamma = -\frac{\partial S}{\partial \mu}$, where $S$ is the action function [8].
A further remark: the dressing matrix approach to Bäcklund transformations, and then to discretization, is closely related to the *factorization method* developed by Moser and Veselov in 1991 [78], [127]. The method is based on the following observations: suppose to have a polynomial Lax matrix with a suitable factorization, say $L(\lambda) = A(\lambda)B(\lambda)$, and suppose that by interchanging the two factor matrices one obtains a

matrix $L'(\lambda)$ belonging to the same polynomial family as $L(\lambda)$. Then the transformed variables appearing in the Lax matrix $L'(\lambda)$ describe the discrete equations of the model under consideration. Actually Moser and Veselov generalized a result of Symes on the Toda chain [119]: he showed how the application of the above process to $L = exp(K)$, where $K$ is a symmetric tridiagonal matrix, leads to an integrable mapping which is interpolated by the Toda flow. From my point of view however everything becomes clearer by thinking at the transformation linking $L(\lambda)$ and $L'(\lambda)$ as a similarity transformation; in fact $L'(\lambda) = B(\lambda)A(\lambda) = A^{-1}(\lambda)L(\lambda)A(\lambda)$, so that the matrix $A(\lambda)$ represents our dressing matrix and all the considerations of this section can be applied to the factorization method.
At this point we have seen the following features of Bäcklund transformations:

**1** They are canonical maps that preserve the algebraic form of the integrals.

**2** They can be constructed by means of a similarity transformation with a dressing matrix $D$. They can depend, in general, by a set of essential parameters $\{\mu_i\}_{i=1}^N$.

**3** It is possible to parametrize the dressing matrix in such a way that its determinant is zero for an arbitrary value of the spectral parameter, say for $\lambda = \mu$. Then the Bäcklund transformations will depend on this parameter. Furthermore it is canonically conjugated to $\gamma$, where $\gamma$ is linked to $\mu$ by the characteristic curve that appears in the linearization of the integrable system. This is the *spectrality property*.

**4** They discretize a family of flows; the interpolating Hamiltonian can depend on a parameter or a set of parameters: actually it is possible that one can choose their value so to obtain the discretized flow of each of the Hamiltonian $H_i$ of the model. How we will see, this is the case for the Gaudin models and the Kirchhoff top.

At this list it is possible to add other very significant features that however are consequences of the previous ones. That is:

**5** Commutativity. Obviously one can compose two Bäcklund transformations with different parameters, say $\mu_1$ and $\mu_2$. Let me symbolize the transformation with parameter $\mu_1$ with $B_{\mu_1}$ and the one with parameter $\mu_2$ with $B_{\mu_2}$. As shown by Veselov [127], by the property of canonicity and invariance of Hamiltonians follows the commutativity of the transformations: $B_{\mu_1}B_{\mu_2} = B_{\mu_2}B_{\mu_1}$. In fact, consider the Poisson map defined by the Bäcklund transformations. If $M_H$ is the manifold of the level set of the integrals:

$$M_H = \{x : H_i(x) = h_i, i = 1 \dots N\},$$

then, if it is compact and connected, for the same reason as in the usual Liouville theorem, it must be a torus $\mathbb{T}^N = \frac{\mathbb{R}^N}{\mathbb{Z}^N}$. Veselov shows that the map determines a shift on this torus; that is for some function $b$

$$\phi_i \to \phi_i + b_i(\mu), \qquad \phi_i \in \mathbb{T}^N.$$

The commutativity is then obvious. It is also possible to take another perspective to explain the Veselov observation. In fact, if the dynamical variables are $\{q_i, p_i\}$ and the transformed variables are $\{\tilde{q}_i, \tilde{p}_i\}$, then, because the transformations are canonical, the new flows can be written again in Hamiltonian form. Because the generating function of the transformations, say the function $F(q, \tilde{q})$ such that $p_i = \frac{\partial F}{\partial q_i}$ and $\tilde{p}_i = -\frac{\partial F}{\partial \tilde{q}_i}$, does not depend on time, there is the equivalence between the two Hamiltonians: $H(q,p) = \tilde{H}(\tilde{q}, \tilde{p})$; but the Bäcklund transformations preserve also the *algebraic* form of the integrals, so that also the algebraic form of $\tilde{H}(\tilde{q}, \tilde{p})$ is preserved: this means that $(q,p)$ and $(\tilde{q}, \tilde{p})$ satisfy the same equations of motion but with different initial conditions (by the contrary one has the trivial Bäcklund $\tilde{q} = q, \tilde{p} = p$ by uniqueness theorem on the solutions of ODEs): the two solutions are related by a shift in time. It is clear now that, regarding the transformed variables $\tilde{q}$ and $\tilde{p}$ as algebraic expressions in terms of $q$ and $p$, they are truly discretization of the dynamical system. A simple example I hope will clarify this point. Let me take an harmonic oscillator with Hamiltonian:

$$H = \sum_i \frac{1}{2}(q_i^2 + p_i^2).$$

One can verify that a family of Bäcklund transformations on this system is given by:

$$(\tilde{q}_i, \tilde{p}_i) = B_\lambda(q,p) = \left( \frac{q_i(1-\lambda^2) - 2p_i\lambda}{1+\lambda^2}, \frac{p_i(1-\lambda^2) + 2q_i\lambda}{1+\lambda^2} \right). \tag{1.59}$$

In fact, if $(q_i(t), p_i(t))$ is a solution of the equations of motion, so is $(\tilde{q}_i(t), \tilde{p}_i(t))$. Obviously for these transformations one has $\{\tilde{q}_i, \tilde{p}_i\} = \{q_i, p_i\}$ and $\tilde{H}_i = \tilde{q}_i^2 + \tilde{p}_i^2 = q_i^2 + p_i^2 = H_i$. Equations (1.59) define a shift: for example by taking $\lambda = 1$, the value of $(\tilde{q}, \tilde{p}) = (-p_i, q_i)$ is the value of the flow at the time $t = \frac{\pi}{2}$. In order to see this, it suffices to re-parametrize the Bäcklund parameter as $\lambda = \tan(\frac{t}{2})$. The transformations now read:

$$\begin{aligned} \tilde{p}_i &= p_i \cos(t) + q_i \sin(t), \\ \tilde{q}_i &= q_i \cos(t) - p_i \sin(t), \end{aligned}$$

that is the general solution of the equations of motion. These ideas will be further developed in the discussion of the Kirchhoff top.

A last observation for this item. As observed in [64], composing $N$ Bäcklund transformations with parameters $\mu_1 \ldots \mu_N$, where $N$ is the number of degree of freedoms of the finite dimensional system, in the angle coordinates one has a shift $\phi_i \to \phi_i + b_i(\mu_1) + \ldots + b_i(\mu_N)$. For generic $b_i$ one has a complete cover of the $N$-dimensional Liouville torus. In this sense the Bäcklund transformations constructed are the most general, any other canonical transformation preserving the integrals must be expressible in terms of $B(\mu_1) \circ B(\mu_2) \circ \ldots \circ B(\mu_N)$.

**6** Explicitness. A powerful consequence of the spectrality property is that the maps constructed are explicit. In fact, as explained before, the kernel of the dressing matrix is the eigenvector of the Lax matrix $L$ calculated at the value $\lambda = \mu$. This gives a direct link between the elements of the dressing matrix and the untilded dynamical variables *only*. Thanks to $\tilde{L} = DLD^{-1}$, it is then possible to make explicit the dependence of the tilded variables by the untilded ones. By a computational point of view the maps are just an iterative process.

**7** The transformed variables can be rational functions of the original ones. More precisely, if the dynamical variables of the system enter linearly in the elements of the Lax matrix (as is the case for the Gaudin and Kirchhoff models), because the Bäcklund transformations are explicitly given by $\tilde{L} = DLD^{-1}$, the only irrationality can arise from the function $\gamma$, that enters in the transformations due to the spectrality property. In fact in general it satisfies an algebraic equation: $\det(L - \gamma \mathbb{1}) = 0$. But $\gamma$ is a generating function for the integrals of the model, so, by fixing the initial condition, it is a constant. The transformations are then rational.

## 1.6 Outline of the Thesis

The order that I have chosen for the chapters represents an attempt to give a rational settlement to the topics treated. It corresponds also to the chronological order in which they occurred during my daily work in these three years of doctoral studies. Apart the above introduction, there are other three chapters and three appendixes supporting the text. In Chapter 2 I introduce the Gaudin models by various points of view, both on the physical and mathematical ground. They are indeed deep connected with the BCS theory of superconductivity and, more in particular, with the Richardson's model. So in the first section of the chapter I briefly review these connections. The Richardson's work probably gave the input to Gaudin to write down his intuitions, so in the second section I review the original work of Gaudin on the model carrying his name. In the third section I show how to obtain by a limiting procedure on the Heisenberg magnet on the lattice, the Lax matrix and the $r$-matrix structures of the Gaudin magnets;

the arguments are due principally to Sklyanin. Furthermore I show also how to every solution of the classical Yang-Baxter equation it corresponds a generalized Gaudin model associated with any semi-simple Lie algebra $\mathfrak{g}$. All the section is supplemented with the appendix B, where I collect the main features of the $r$-matrix formalism. Finally in the last part I give a very brief review of the various applications that until now, at my knowledge, the Gaudin models found in the scientific world. The fourth section of the chapter is devoted to the so called "algebraic extension" of Gaudin models: essentially through an Inönü Wigner contraction on the Lie algebra underlying the models and an associated procedure of *pole coalescence* on the Lax matrix, one is able to recover new, but sometimes known, integrable systems and their related integrability structures. A practical application of the material contained in this section will be given in chapter 4. The chapter 3 is the core of the entire work. There was the possibility to reverse the order of the sections in this chapter, from the more general elliptic case to the simpler rational case, but I prefer to present again the arguments in chronological order also because the trigonometric case has been fundamental to obtain more insight into the elliptic case. For the rational case, given in the first section, the most of the results shown appeared in [52] and [111]. Furthermore I write also the explicit two-points maps showing how the transformations allow, starting with a *real* solution of the equations of motion, to obtain new *real* solutions. This result is however a simple consequence (or limit) of the analogue properties in the more general trigonometric and elliptic cases. The second section is the result of a work of O. Ragnisco and me [93], [94]. Here I generalize the rational construction to the trigonometric case in the most natural way, showing also how all the rational results appear in the limit of "small angles". All the beautiful properties, such explicitness, symplecticity, spectrality, limits to continuous flows, preservation of integrals of the continuous flows that the map discretizes and transformation between real solutions of the equations of motion are all kept safe. The last section is obviously devoted to the elliptic case. The results are mine and, so far, are unpublished. Also in this case it is possible to repeat all the observations as for the previous section. The appendix (A) clarifies the notations used for the elliptic functions and contains some lengthy proofs. The last chapter contains an application of the results given in the second section of chapter 3 to the Kirchhoff top. This model is the algebraic extension of the trigonometric Gaudin model [80] [86]. The results are been obtained by O. Ragnisco and myself. Some of them have been published [95], some others, like the separation of variables, are new. In the last section of the chapter I will give an explicit example of one of the ideas that enter repeatedly in the thesis, i.e. that the Bäcklund transformations for finite dimensional system can be not only a theoretical, but indeed also a practical tool to obtain the *general solution* of the equations of motion. As I will show, this is connected to the fact that the generating function of the integrals of the system can be interpreted as the action (the Hamilton's principal function) that leads to the Hamilton-Jacobi separability of the system and furthermore turns out to be also the interpolating Hamiltonian flow of the discrete dynamics generated by the Bäcklund transformations. As a matter of fact, by a practical point of view, a re-parametrization of the Bäcklund parameter playing the role of the time step in discrete dynamics, leads to a continuous flow that is exactly the general solution of the continuous model. This can be better understood by thinking at the Veselov's observation [127] that a canonical map preserving the integrals of motion, as did the Bäcklund transformations, is a shift on the Liouville torus.

# Chapter 2

# The Gaudin models

## 2.1 A short overview on the pairing model

The Gaudin models describe completely integrable long range spin-spin systems. Their particular characteristics hold both at the classical and at the quantum level. Originally they were introduced by Gaudin [43]. His work was surely influenced by those of Baxter [11], Yang [130], Sutherland [117] Richardson and Sherman [99] and Richardson [98] on exact solutions of systems of interacting spins and was built on his previous knowledge [44] of the Bethe method for solving one-dimensional problems. Can be instructive to give a brief overview of Richardson's construction of solutions of the so called pairing model [98] (see also [5] or [32] for a review; for more detailed accounts the reader is referred to [99] and [98]; for an historical overview and considerations from several viewpoints of BCS superconductivity theory and its links with Gaudin magnet see [105]).
The physical system is composed by $N$ interacting fermions, whose dynamic is governed by a pairing Hamiltonian consisting of the kinetic term and an interaction which describes an attraction between electrons:

$$H = \sum_{j} \epsilon_{j\sigma} c^{\dagger}_{j\sigma} c_{j\sigma} - g \sum_{j,j'} c^{\dagger}_{j\uparrow} c^{\dagger}_{j\downarrow} c_{j'\downarrow} c_{j'\uparrow}, \tag{2.1}$$

where $\sigma \in \{\uparrow, \downarrow\}$, $j \in 1 \dots \Omega$, $c_{j,\sigma}$ annihilates a particle in the state $(j\sigma)$, $c^{\dagger}_{j,\sigma}$ creates a particle in the state $(j\sigma)$ and $g$ is the coupling constant. The sums run over a set of doubly degenerate energy levels $\epsilon_j$ $(j = 1 \dots \Omega)$. In the course of time a series of exact results have been obtained for the quantum problem defined by (2.1). Noteworthy is the work of Richardson and Sherman [99] in this sense. More recently the integrals of motion and the diagonalization by means of algebraic Bethe ansatz were obtained in [107] and [28].
A generic physical state is composed by $M$ Cooper pairs and $v$ unpaired particles, so

that $N = 2M + v$. The single occupied states are *frozen* so that one can focus on the $M$ pairs [5]. Note that it is possible to use a spin realization of pair operators: by rewriting the Hamiltonian (2.1) in the new variables one obtains a quantum spin model with long range interaction. It is this observation that gives the link between pairing models and Gaudin systems. In particular, if $s_j^- \doteq c_{j\downarrow}c_{j\uparrow}$, $s_j^+ \doteq \left(s_j^-\right)^\dagger = c_{j\uparrow}^\dagger c_{j\downarrow}^\dagger$ and $2s_j^z + 1 \doteq c_{j\uparrow}^\dagger c_{j\uparrow} + c_{j\downarrow}^\dagger c_{j\downarrow}$, then the new variables, due to the anticommutation relations between the fermionic operators, are the spin-$\frac{1}{2}$ realization of $\mathfrak{su}(2)$. The Hamiltonian (2.1) in these new variables reads:

$$H = \sum_j \epsilon_j s_j^z - \frac{g}{2}\sum_{j,j'}(s_j^+ s_{j'}^- + s_j^- s_{j'}^+),$$

where constant factors have been omitted. Richardson [99] made this ansatz for the Cooper eigenstates $H|M\rangle = E|M\rangle$:

$$|M\rangle = \prod_{k=1}^{N} B_k|0\rangle \qquad \text{with} \qquad B_k \doteq \sum_j \frac{s_j^+}{2\epsilon_j - E_k},$$

where $|0\rangle$ is the lowest weight vector $|1/2, -1/2\rangle$. It is possible to show [5], [98], [99] that indeed the previous is an eigenvector provided the "Bethe" or "Richardson" equations for $E_k$'s are fulfilled:

$$\frac{1}{g} - \sum_l^{\Omega} \frac{1}{2\epsilon_l - E_k} + \sum_{j\neq k}^{M} \frac{2}{E_j - E_k} = 0.$$

A set of involutive integrals for this model is given by (see [28]):

$$\tau_j = \frac{1}{g}s_j^z - \sum_{j'\neq j} \frac{\vec{s}_j \cdot \vec{s}_{j'}}{\epsilon_j - \epsilon_{j'}}, \qquad [H, \tau_j] = 0, \qquad [\tau_j, \tau_k] = 0, \tag{2.2}$$

where

$$\vec{s}_j = \left(s_j^x, s_j^y, s_j^z\right), \qquad \begin{cases} s_j^x + is_j^y \doteq s_j^+ \\ s_j^x - is_j^y \doteq s_j^-. \end{cases}$$

Note that the pairing Hamiltonian can be expressed as a function of the integrals [5]:

$$\frac{1}{g^3}H = \frac{1}{g^2}\sum_j \epsilon_j \tau_j + \sum_{j,j'} \tau_j \tau_{j'},$$

where again additive constants have been omitted. At this point, for those who yet know the Gaudin models, should be clear that in the limit $g \to \infty$ one obtains just the rational Gaudin model.

## 2.2 The Gaudin generalization

When Gaudin wrote his article [43] was aware of the Richardson results on the pairing Hamiltonian solution. Given the operators:

$$H_j = \sum_{j' \neq j} \frac{\vec{s}_j \cdot \vec{s}_{j'}}{\epsilon_j - \epsilon_{j'}}, \tag{2.3}$$

he chose to consider the more general form:

$$H_j = \sum_{\substack{j' \neq j \\ j=1}}^{N} \sum_{\alpha=1}^{3} w_{jj'}^{\alpha} \, s_j^{\alpha} s_{j'}^{\alpha}. \tag{2.4}$$

By using his words, the question he posed was [43]:"*Quelles sont les conditions sur les coefficients* $w_{jj'}^{\alpha}$ *pour avoir les commutations* $[H_j, H_k] = 0$?". From the commutation relations between the spin operators one readily finds the answer; the necessary and sufficient conditions are expressible in terms of the following relations on the coefficients $w_{jj'}^{\alpha}$:

$$w_{ij}^{\alpha} w_{jk}^{\gamma} + w_{ji}^{\beta} w_{ik}^{\gamma} - w_{ik}^{\alpha} w_{jk}^{\beta} = 0, \tag{2.5}$$

to be satisfied for all the permutations of $\alpha, \beta$ and $\gamma$ and with $i, j, k$ distinct. To better manipulate these expressions, a further hypothesis on the antisymmetry of this coefficients was added by Gaudin, that is:

$$w_{ij}^{\alpha} = -w_{ji}^{\alpha}.$$

The antisymmetric relations corresponds, by an operatorial point of view, to the equivalence $[s_i^z, H_j] = [s_j^z, H_i]$. At this point Gaudin noted the formal analogy of the equation (2.5) with the quartic Riemann relations for the Jacobi theta functions (see for example [79] [125]). In particular, by posing:

$$w_{ij}^{\alpha} = \frac{\Theta_{\alpha+1}(u_{ij})}{\Theta_{\alpha+1}(0)\Theta_1(u_{ij})},$$

equations (2.5) become:

$$\Theta_2(u_{ij})\Theta_4(u_{jk})\Theta_1(u_{ki})\Theta_3(0) + \Theta_1(u_{ij})\Theta_3(u_{jk})\Theta_2(u_{ki})\Theta_4(0) + \\ + \Theta_3(u_{ij})\Theta_1(u_{jk})\Theta_4(u_{ki})\Theta_2(0) = 0$$

which suggests the solution:

$$\Theta_\alpha(u_{ij}) = \theta_\alpha \left( \frac{\pi(u_i - u_j)}{2K}, e^{\left(-\pi \frac{K'}{K}\right)} \right),$$

where $K$ and $K'$ are respectively the complete elliptic integral of the first and the complementary integral (see A). In terms of the Jacobi elliptic functions one has (see also [7]):

$$w_{ij}^1 = \frac{1}{\mathrm{sn}(u_i - u_j)}, \qquad w_{ij}^2 = \frac{\mathrm{dn}(u_i - u_j)}{\mathrm{sn}(u_i - u_j)}, \qquad w_{ij}^3 = \frac{\mathrm{cn}(u_i - u_j)}{\mathrm{sn}(u_i - u_j)}, \tag{2.6}$$

where I have omitted for simplicity the elliptic modulus $k$. When the projection of the total spin on the $z$ axis commute with all the operators $H_j$, the solution degenerates into:

$$w_{ij}^1 = w_{ij}^2 = \frac{1}{\sin(u_i - u_j)}, \qquad w_{ij}^3 = \frac{\cos(u_i - u_j)}{\sin(u_i - u_j)}, \tag{2.7}$$

that corresponds to the limit $k \to 0$ in the general result (2.6). The isotropic case in turns is recovered by posing $u_j = \epsilon_j u$ and taking the limit $u \to 0$.
The corresponding involutive integrals are given by:

$$H_j^e = \sum_{k \neq j}^{N} \frac{s_j^z s_k^z \mathrm{cn}(u_j - u_k) + s_j^y s_k^y \mathrm{dn}(u_j - u_k) + s_j^x s_k^x}{\mathrm{sn}(u_j - u_k)}, \tag{2.8}$$

for the elliptic solution (2.6) and

$$H_j^t = \sum_{k \neq j}^{N} \frac{\cos(u_j - u_k) s_j^z s_k^z + s_j^x s_k^x + s_j^y s_k^y}{\sin(u_j - u_k)} \tag{2.9}$$

for the trigonometric solution (2.7) and obviously (2.3) for the isotropic or rational one. Often in the literature one speaks of $XXX$, $XXZ$ and $XYZ$ Gaudin models meaning respectively (2.3), (2.9) and (2.8).
One of the most important features of the Gaudin models is that not only they can be formulated in the $r$-matrix framework (see Appendix B) both at the classical and at the quantum level, but also that this formulation holds for whatever dependence on the spectral parameter.

## 2.3 Lax and $r$-matrix structures.

The Lax and $r$-matrix structures of the Gaudin model can be introduced in various way. Maybe the simplest one is to view them as a limiting case of the corresponding structures of the lattice Heisenberg magnet. It is noteworthy however to note that it is possible to associate a Gaudin model with any semi simple Lie algebra $\mathfrak{g}$; the Poisson structure can be then described in terms of the corresponding $r$-matrix, i.e. the solution of the classical Yang-Baxter equation as given by Belavin and Drinfeld

[14], [15] (see appendi B). In the following I will follow closer the first point of view; anyway the second one is clarified at the end of the section.
The lattice version of the Heisenberg model, the time being continuous, so that one has a chain of interacting spins, is well described in [35]. It was first introduced by Sklyanin [106]. Since at every point of the lattice there corresponds a set of spin dynamical variables, one has a Lax matrix that depends on the site index, $L = L_n(\lambda)$. The Poisson brackets between the elements of the Lax matrices $L_n(\lambda)$ are retrieved from the corresponding continuous model; they are defined by the quadratic $r$-matrix relation:

$$\{L_{1,n}(\lambda), L_{2,m}(\mu)\} = [r(\lambda-\mu), L_{1,n}(\lambda)\otimes L_{2,m}(\mu)], \tag{2.10}$$

where $r(\lambda-\mu)$ is given by the elliptic, trigonometric or rational solution of Belavin and Drinfeld for the classical Yang-Baxter equation (see B); since the trigonometric and rational cases can be obtained by a limiting procedure on the elliptic case, just as explained in the above paragraph for the Gaudin Hamiltonians, in the next rows I will deal only with this last one. The $r$-matrix is then given by:

$$r(\lambda) = \sum_{\alpha=1}^{3} f^\alpha(\lambda)\sigma_\alpha \otimes \sigma_\alpha, \tag{2.11}$$

where the functions $f^\alpha(\lambda)$, $\alpha = 1,2,3$, according to (B) are defined as:

$$f^1(\lambda) = \frac{1}{\mathrm{sn}(\lambda)}, \qquad f^2(\lambda) = \frac{\mathrm{dn}(\lambda)}{\mathrm{sn}(\lambda)}, \qquad f^3(\lambda) = \frac{\mathrm{cn}(\lambda)}{\mathrm{sn}(\lambda)}.$$

With $\sigma_\alpha$ hereafter I denote the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.12}$$

The Lax matrix of the $n^{\mathrm{th}}$ site has the form [35]:

$$L_n(\lambda) = \mathcal{S}_n^0 \mathbb{1} + i\sum_{\alpha=1}^{3} f^\alpha(\lambda)\mathcal{S}_n^\alpha \sigma_\alpha. \tag{2.13}$$

By a direct calculation it is possible to show that the Poisson brackets (2.10) are equivalent to the following brackets for the dynamical variables $\mathcal{S}_n^\alpha$, $\alpha = 0\ldots3$:

$$\begin{aligned} \{\mathcal{S}_n^i, \mathcal{S}_m^j\} &= -\mathcal{S}_n^0\mathcal{S}_n^k\delta_{n,m} \\ \{\mathcal{S}_n^i, \mathcal{S}_m^0\} &= J_{jk}\mathcal{S}_n^j\mathcal{S}_n^k\delta_{n,m}, \end{aligned} \tag{2.14}$$

where $(i,j,k)$ is a cyclic permutation of $(1,2,3)$ and $J_{jk}$ is an antisymmetric tensor explicitly given by:

$$\boldsymbol{J} = \begin{pmatrix} 0 & k^2 & 1 \\ -k^2 & 0 & 1-k^2 \\ -1 & k^2-1 & 0 \end{pmatrix}, \tag{2.15}$$

with $k$ the elliptic modulus of the Jacobi elliptic functions. In proving these formulae, the relations (A.9-A.14) are very useful. The integrals of the system are given by the trace of the powers of the so-called *monodromy matrix* [35], usually denoted with $T(\lambda)$ but that I will indicate with $L(\lambda)$ as it is indeed the Lax matrix for the entire chain. It is given by [55], [108], [109]:

$$L(\lambda) = L_N(\lambda)L_{N-1}(\lambda)\dots L_1(\lambda). \tag{2.16}$$

Notice that the Poisson brackets for the monodromy matrix are the same as for the one point Lax matrix thanks to the ultralocal nature of (2.10) (the elements of the Lax matrices that belong to different sites are in involution). It is possible to extract a physical Hamiltonian by the trace of the monodromy matrix: taking a point, say $\lambda_0$, at which $L_n(\lambda)$ degenerates, that is $\det(L_n(\lambda_0) = 0)$, one defines [35]:

$$H = \log\left(\frac{|\mathrm{tr}L(\lambda_0)|^2}{2}\right) = \sum_{n=1}^{N} \log\left(\mathcal{S}_n^0\mathcal{S}_{n+1}^0 + \sum_{a=1}^{3} c_a\mathcal{S}_n^a\mathcal{S}_{n+1}^a\right), \tag{2.17}$$

where $c_a$ is a constant depending on the two Casimirs of the brackets (2.14) and on the elliptic modulus $k$. The corresponding equations of motion are given by:

$$\frac{d\mathcal{S}_n^\alpha}{dt} = \{H, \mathcal{S}_n^\alpha\}, \qquad a = 0\dots 3.$$

For a derivation of solitonic solutions in the trigonometric case ($k = 0$) of these equations, see [121]. The Hamiltonian (2.17) is physical in the sense that one recovers the continuous Heisenberg chain with the usual limiting procedure, i.e. by posing $nh = x$ and taking $h \to 0$. In fact, defining the set $J_i$, $i = 1, 2, 3$ as given by $4J_{ij} = J_j - J_i$, where $J_{ij}$ is the antisymmetric tensor (2.15), one finds the continuous Hamiltonian as [35]:

$$-4H + 4N\mathrm{log}2 = h\sum_{a=1}^{3}\int\left(\left(\frac{d\vec{S}^a(x)}{dx}\right)^2 - J_aS^a(x)^2\right)dx + O(h^2).$$

The previous equation is obtained by considering that in the limit $h \to 0$, $\mathcal{S}_n^0 \to 1 + O(h^2)$ and $\mathcal{S}_n^a \to hS^a(x)$, $a = 1, 2, 3$. The equations of motion obviously turn to be the Landau-Lifschitz equations:

$$\frac{\partial\vec{S}}{\partial t} = \vec{S}\wedge\frac{\partial^2\vec{S}}{\partial x^2} + \vec{S}\wedge J\vec{S},$$

with $J\vec{S} \doteq (J_1(S^1)^2, J_2(S^2)^2, J_3(S^3)^2)$.
Sklyanin showed [109] how to obtain the integrability structures of Gaudin model, that is the Lax matrix and the $r$-matrix, by a limiting procedure on the lattice Heisenberg

model. One has to introduce a small parameter, say $\epsilon$, and to take $r \to i\epsilon r$ in (2.11) and $\mathcal{S}_n^i \to -i\epsilon s_n^i \mathcal{S}_n^0$ in (2.13). Because the $r$-matrix depends only on the difference $\lambda - \mu$, it is possible to add a shift in the spectral parameter of every Lax matrix defined on a point of the lattice. This does not alter the structure (2.10). So one is left with the one point Lax matrix (cf. 2.13):

$$L_n(\lambda) = \mathcal{S}_n^0 \left( \mathbb{1} + \epsilon \sum_{\alpha=1}^{3} f^\alpha(\lambda - \lambda_n) s_n^\alpha \sigma_\alpha \right). \tag{2.18}$$

The Poisson brackets (2.14) degenerate into:

$$\begin{aligned} \{s_n^i, s_m^0\} &= O(\epsilon^2), \\ \{s_n^i, s_m^j\} &= s_n^k \delta_{n,m}, \end{aligned} \tag{2.19}$$

so that, for $\epsilon \to 0$ and for every point $n$ of the lattice, the dynamical variables reduce to the triple $(s_n^1, s_n^2, s_n^3)$. At first order in $\epsilon$ it is simple to show that the monodromy matrix (2.16) goes into:

$$L(\lambda) = \prod_k L_k(\lambda) \Rightarrow \left( \prod_k \mathcal{S}_k^0 \right) \left( 1 + \epsilon \sum_{k,\alpha} f^\alpha(\lambda - \lambda_k) s_k^\alpha \sigma_\alpha \right) \doteq \left( \prod_k \mathcal{S}_k^0 \right) (1 + \epsilon L_{\text{Gaudin}}),$$

where $L_{\text{Gaudin}}$ identifies the Lax matrix for the Gaudin chain:

$$L_{\text{Gaudin}} = \sum_{k=1}^{N} \begin{pmatrix} s_k^3 f^3(\lambda - \lambda_k) & s_k^1 f^1(\lambda - \lambda_k) - i s_k^2 f^2(\lambda - \lambda_k) \\ s_k^1 f^1(\lambda - \lambda_k) + i s_k^2 f^2(\lambda - \lambda_k) & -s_k^3 f^3(\lambda - \lambda_k) \end{pmatrix}. \tag{2.20}$$

The functions $f^1(\lambda)$, $f^2(\lambda)$ and $f^3(\lambda)$ are respectively given by (B.8), (B.9) and (B.10). Note that the Poisson brackets (2.10) becomes *linear* in the limit $\epsilon \to 0$:

$$\{L_{\text{Gaudin},1}(\lambda), L_{\text{Gaudin},2}(\mu)\} = [r(\lambda - \mu), L_{\text{Gaudin},1}(\lambda) + L_{\text{Gaudin},2}(\mu)]. \tag{2.21}$$

With some calculations it is possible to show that indeed the previous brackets entail (2.19). Hereafter the subscript "Gaudin" below the letter $L$ will be omitted for convenience.

Now, following [57], [80], [86], [96], [115], I will show how to define a generalized Gaudin model for every solution of the classical Yang Baxter equation. With generalized I mean that the model is associated with any semi simple Lie algebra $\mathfrak{g}$ and not just with $\mathfrak{sl}(2)$. Consider the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ with the contraction between the two respective basis, $\{X_\alpha\}$ and $\{X^\alpha\}$, $\alpha = 1 \ldots \dim\mathfrak{g}$, given by $\langle X^\alpha, X_\beta \rangle = \delta^\alpha_\beta$. On $\mathfrak{g}^*$ is defined a natural Poisson structure [74], [120], [124]; let $F$ and $G$ be two functionals on $\mathfrak{g}^*$, so that $F, G : \mathfrak{g}^* \to \mathbb{C}$ and define, for $L \in \mathfrak{g}^*$, the bracket between $F$ and $G$ as:

$$\{F, G\}(L) \doteq \langle L, [\frac{\delta F}{\delta L}, \frac{\delta G}{\delta L}] \rangle, \tag{2.22}$$

where $\frac{\delta F}{\delta L}$ is defined by $\frac{d}{d\epsilon}F\left(L+\epsilon\delta L\right)\Big|_{\epsilon=0} \doteq \langle \delta L, \frac{\delta F}{\delta L}\rangle$. It is easy to verify that the definition (2.22) satisfies the Jacobi identity. Now take an arbitrary $L \in \mathfrak{g}^*$, say $L = s^\alpha X_\alpha$; the component $s^\alpha$ of the coordinates on $\mathfrak{g}^*$ is given by $s^\alpha = \langle L, X^\alpha\rangle$. The gradient of $s^\alpha$ is $X^\alpha$ and in local coordinates one has:

$$\{s^\alpha, s^\beta\}(L) = \langle L, [X^\alpha, X^\beta]\rangle = c_\gamma^{\alpha\beta}\langle L, X^\gamma\rangle = c_\gamma^{\alpha\beta}s^\gamma. \tag{2.23}$$

Now define the following matrix:

$$\ell(\lambda) = g_{\alpha\beta}X^\alpha s^\beta f^\alpha(\lambda-\omega), \tag{2.24}$$

By a direct check it is possible to show [80] that the matrix $\ell$ satisfies the linear $r$-matrix structure (2.21): in fact inserting (2.24) in (2.21), one reduces to the system (B.7) given in appendix B. Notice that, as yet explained some lines ago, the shift of the spectral parameter does not alter the Poisson structure because the $r$-matrix depends only on the difference of its arguments. The Gaudin models are now obtained by considering $N$ copies of $\ell$. Formally one must consider the semi simple algebra given by the direct sum $\oplus^N\mathfrak{g}$, the corresponding basis given by $X_i^\alpha$, $\alpha = 1\ldots\dim\mathfrak{g}$, $i = 1\ldots N$, the Lie bracket on $\oplus^N\mathfrak{g}$:

$$[X_i^\alpha, X_j^\beta] = c_\gamma^{\alpha\beta}X_i^\gamma\delta_{ij}$$

and the corresponding dual algebra with its basis. If $\{s_i^\alpha\}_{\alpha=1}^{\dim\mathfrak{g}}$ are the coordinates on $\mathfrak{g}^*$, the Lax matrix for the generalized Gaudin models is finally given by:

$$L(\lambda) = \sum_{i=1}^N g_{\alpha\beta}X^\alpha s_i^\beta f^\alpha(\lambda-\lambda_i). \tag{2.25}$$

In the $\mathfrak{su}(2)$ case, by taking the Pauli matrices (2.12) as a basis of its fundamental representation, one obtains the result as given in (2.20).
From the point of view of separation of variables, functional Bethe ansatz and quantum inverse scattering method, a lots of works on Gaudin models began to appear until 80s and goes on even now [3], [33], [34], [45], [57], [58], [107], [113], [114]. In 2001 the Bäcklund transformations for the rational Gaudin model (in the $\mathfrak{su}2$ case) and the related integrable discretization were constructed by Hone, Kuznetsov and Ragnisco [52]. The same result appear independently in a pair of papers by Sklyanin in 2000 [110], [111]. In the first of these papers Sklyanin made a conjecture about the possibility to extend these results to the trigonometric and elliptic case, but the question remained pending until now. One of the first aims of this work is to fill the gap.
In the last twenty years a great number of works have pointed up, quite surprisingly, very interesting connections between the Gaudin models and various branches of mathematics and physics. A complete list of the papers that have appeared in this sense is rather impossible. A recent collection of the main research areas in which Gaudin

models enter significantly can be found in [90]. Just to give an idea of the heterogeneity of the subjects involved, I can say for example that in [41] Frenkel shows how the equivalence between two different realizations of the *Langlands correspondence* for the group $\mathfrak{sl}(2)$ amounts to a separation of variables in the rational Gaudin model; in [49] and [50] Hikami et al. gave an integral representation for solutions of the Knizhnik-Zamolodchikov equations by using the results obtained with the quantum inverse scattering method on the trigonometric Gaudin model; the integrability of the Seiberg-Witten theory of supersymmetric gauge theory is deeply connected with the integrability structure of the elliptic Gaudin model [46]; there are possible practical and surely theoretical applications to BCS theory [99] and small metallic grains [5], [28]; there are fruitful connections with pairing models in nuclear physics [31], [32]; there are integrable systems connected to the Coulomb three body problem that are indeed four-sites rational Gaudin model [60]; and finally, there are classical integrable systems, as Lagrange tops or the Kirchhoff equations, that arise by the Gaudin models through a contraction procedure on the underlying Lie algebra [80], [86], the so called Inönü Wigner contraction (more precisely a generalized Inönü Wigner contraction). In some sense the Gaudin models are so robust that all the results on Bäcklund transformations can be applied, with some care, to the interacting systems originating from this contraction procedure. As an application of this idea I will give in the last chapter a series of new results on the Kirchhoff top. This is why in the next section I will clarify, following [80], how to *contract* the Gaudin systems so to obtain new integrable systems and their associated integrable structures.

## 2.4 Inöñu-Wigner contraction and poles coalescence on Gaudin models

In 1953 Inönü and Wigner [56], following a work of Segal [103], ask in what sense groups can be limiting cases of other groups. They were thinking at the Galilei group, the group of classical mechanics, that must be a limiting case of the Poincarè group, the group of isometries of Minkowski spacetime. In their paper, the two authors introduced the so-called simple Inönü Wigner contractions; their formulation was then extended in a more general setting [123]. In this section, for the theoretical concepts about contractions of groups I will follow [123], for the direct application of these concepts to the Gaudin models I will follow [80].
Let $\mathfrak{L} = (S, \mu)$ be a Lie algebra, where $S$ is the underlying vector space and $\mu : S \times S \to S$ represents the Lie multiplication. Consider a continuous family of linear mappings $U(\epsilon) : S \to S$ that are nonsingular for $\epsilon > 0$ and singular for $\epsilon = 0$. For $\epsilon > 0$ let

$$\mu_\epsilon(X^a, X^b) = U^{-1}(\epsilon)\mu\left(U(\epsilon)X^a, U(\epsilon)X^b\right), \qquad X^a, X^b \in S$$

be the new Lie bracket defining the new Lie Algebra $\mathfrak{L}_\epsilon = (S, \mu_\epsilon)$, isomorphic to $\mathfrak{L}$. If the limit:

$$\mu' = \lim_{\epsilon \to 0} \mu_\epsilon(s_a, s_b)$$

exists for any $X^a, X^b \in S$, then the Lie algebra $\mathfrak{L}' = (S, \mu')$ will be the contraction of $\mathfrak{L}$. Notice that in general $\mathfrak{L}'$ will not be isomorphic to $\mathfrak{L}$.
A generalized contraction is obtained on the direct sum of subspaces:

$$S = \bigoplus_{i=1}^{N} S_i,$$

with the action of $U(\epsilon)$ on each subspaces $V_i$ being defined by:

$$U(\epsilon)V_i = \epsilon^{n_i} V_i,$$

with $0 \leq n_0 < n_1 \ldots < n_N$, $n_i \in \mathbb{R}$. The Lie algebra $\mathfrak{L} = (S = \bigoplus_{i=1}^{N} S_i, \mu)$ admits the generalized contraction iff [123]:

$$\mu(S_i, S_j) \subset \bigoplus_k S_k,$$

where the sum is over all $k$ such that $n_k \leq n_i + n_j$. The contracted Lie algebra $\mathfrak{L}' = (S, \mu')$ is given by [123]:

$$\mu'(S_i, S_j) \subset S_k, \qquad n_k = n_i + n_j,$$

where all the surviving structure constants are the same as for $\mathfrak{L}$. Otherwise $\mu'(S_i, S_j) = 0$. So in general the contracted Lie algebra $\mathfrak{L}' = (S, \mu')$ is "more abelian" than $\mathfrak{L}$; the interesting contractions will lie between the two trivial ones, $\mathfrak{L}' = L$ and $\mathfrak{L}'$ abelian, that is $\mu' = 0$.
For the Gaudin models one has $N$ copies of $\mathfrak{su}(2)$ spin algebra, or, more in general, $N$ copies of a simple Lie algebra $\mathfrak{g}$: $\mathfrak{L} = \oplus^N \mathfrak{g}$. Let the set $\{X_i^\alpha\}_{\alpha=1}^{\dim\mathfrak{g}}$ denotes a basis of the $i^{\text{th}}$ copy of $\mathfrak{g}$; the Lie multiplication is defined by the usual commutators:

$$[X_i^\alpha, X_j^\beta] = c_\gamma^{\alpha\beta} X_i^\gamma \delta_{ij},$$

where $c_{\alpha\beta}^\gamma$ are the structure constants. The action of the isomorphism $U(\epsilon)$ on the subspaces $\{X_i^\alpha\}_{\alpha=1}^{\dim\mathfrak{g}}$ is defined as [80]:

$$\begin{pmatrix} Y_0^\alpha \\ \vdots \\ Y_{N-1}^\alpha \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \epsilon\lambda_1 & \epsilon\lambda_2 & \cdots & \epsilon\lambda_N \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon^{N-1}\lambda_1^{N-1} & \epsilon^{N-1}\lambda_2^{N-1} & \cdots & \epsilon^{N-1}\lambda_N^{N-1} \end{pmatrix} \begin{pmatrix} X_1^\alpha \\ \vdots \\ X_N^\alpha \end{pmatrix}, \tag{2.26}$$

where $\lambda_i$ are a set of $N$ distinct arbitrary complex parameters. The new commutators $\mu_\epsilon$ are then given by:

$$[Y_i^\alpha, Y_j^\beta]_\epsilon = \sum_{m,n=1}^{N} \epsilon^{i+j} \lambda_m^i \lambda_n^j [X_m^\alpha, X_n^\beta] = c_\gamma^{\alpha\beta} \epsilon^{i+j} \sum_{n=1}^{N} \lambda_n^{i+j} X_n^\gamma = \begin{cases} c_\gamma^{\alpha\beta} Y_{i+j}^\gamma & i+j < N \\ O(\epsilon^N) & i+j \geq N. \end{cases}$$

The limit $\epsilon \to 0$ is well defined and the contracted algebra is:

$$[Y_i^\alpha, Y_j^\beta] = \begin{cases} c_\gamma^{\alpha\beta} Y_{i+j}^\gamma & i+j < N \\ 0 & i+j \geq N. \end{cases} \tag{2.27}$$

As seen in the previous section, on the dual of the Lie algebra $\mathfrak{L} = \oplus^N \mathfrak{g}$ it is naturally defined a Poisson structure; it is reasonable to ask what is the corresponding Poisson structure induced by the map $U(\epsilon)$. The answer is straightforward: in fact it produces on the coordinate set $\{s_i^\alpha\}_{\alpha=1}^{\dim\mathfrak{g}}$, $i = 1 \dots N$ the following isomorphism:

$$S_i^\alpha \doteq \epsilon^i \sum_{k=1}^{N} \lambda_k^i s_k^\alpha, \qquad 0 \leq i \geq N-1.$$

Since the Poisson brackets between the $s_i^\alpha$ are inherited by the Lie multiplications among the basis elements of the Lie algebra $\mathfrak{L}$ (see (2.23)), in the limit $\epsilon \to 0$ the Lie Poisson brackets for the new coordinates $\{S_i^\alpha\}_{\alpha=1}^{\dim\mathfrak{g}}$ are given by:

$$[S_i^\alpha, S_j^\beta] = \begin{cases} c_\gamma^{\alpha\beta} S_{i+j}^\gamma & i+j < N \\ 0 & i+j \geq N. \end{cases} \tag{2.28}$$

The key point is now to extend the contraction at the level of the Lax and $r$-matrix structures; the main features of the corresponding construction as given in [80] can be summarized as follows:

- Consider the Lax matrix of the generalized Gaudin models (2.25) with the associated linear $r$-matrix structure:

$$L(\lambda) = \sum_{i=1}^{N} g_{\alpha\beta} X^\alpha s_i^\beta f^\alpha(\lambda - \lambda_i).$$

- Make the substitutions $\lambda_i \to \epsilon\lambda_i$.
- Take the series expansion in $\epsilon$.
- Take the limit $\epsilon \to 0$ (the *poles coalescence*).

- The Lax matrix $L'$ obtained satisfies the linear $r$-matrix structure:

$$\{L'_1(\lambda), L'_2(\mu)\} = [r(\lambda-\mu), L'_1(\lambda) + L'_2(\mu)] \tag{2.29}$$

with the same $r$-matrix. More specifically $L'$ is explicitly given by:

$$L'(\lambda) = \sum_{i=0}^{N-1} g_{\alpha\beta} X^\alpha S_i^\beta f_i^\alpha(\lambda),$$

where $f_i^\alpha(\lambda)$'s are equal to $\frac{(-1)^i}{i!}\left(\frac{d}{d\lambda}\right)^i f^\alpha(\lambda)$, with $f^\alpha$'s any of the solutions of the system (B.7), given, in the case $\mathfrak{g} = \mathfrak{su}(2)$ by (B.8), (B.9) and (B.10). The $r$-matrix structure (2.29) is equivalent to the Poisson brackets (2.28) for the dynamical variables $S_i^\alpha$, $\alpha = 1\dots \dim\mathfrak{g}$, $i = 1\dots N$.

The proof of these statements is simple; after the substitution $\lambda_i \to \epsilon\lambda_i$ one has to take the formal series expansion in $\epsilon$ in the formula (2.25), make the identification $S_i^\alpha = \epsilon^i \sum_{k=1}^N \lambda_k^i s_k^\alpha$, take the limit $\epsilon \to 0$ and obtain $L'(\lambda)$. The $r$-matrix structure is then directly checked with the help of the system (B.7). The interested reader is referred to [80] for all the details. Notice that (2.29) describes a one-body dynamical system with $N$ degrees of freedom; the extension of this Lax matrix to the many-body case is straightforward [89]: one has only to consider the direct sum of $M$ copies of the model, just as done in equation (2.25) for the generalized Gaudin models. The simplest of the extended models that it is possible to obtain with the contraction procedure are provided taking $\mathfrak{g} = \mathfrak{su}(2)$ and fixing $N = 2$ and $M = 1$. Since the contraction of $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ gives the Euclidean algebra $\mathfrak{e}(3)$ [80], it is natural that the physical contents of this model includes or a set of three space coordinates and three angular momenta or three velocities and three angular momenta. Indeed in the rational $r$-matrix case one obtains the Lagrange top, whereas in the trigonometric and elliptic cases particular realizations of the Kirchhoff model [89], describing the motion of a rigid body in an incompressible fluid.
A last remark. If one is able to construct the Bäcklund transformations for the Gaudin magnets, he then obtains, *mutatis mutandis*, also the Bäcklund transformations for their algebraic extensions. In fact the dressing matrix will not depend, in general, by the arbitrary parameters $\lambda_i$ appearing in the Gaudin models so that the *poles coalescence* does not affect it. This idea has been used in [63] to construct the Bäcklund transformations for the Lagrange top, having those for the rational Gaudin magnet. In the last chapter I will use the results for the trigonometric Gaudin magnet to construct the transformations for the Kirchhoff top.

# Chapter 3

# Bäcklund transformations on Gaudin models

This chapter is the core of the entire work. It can be considered the natural continuation, about ten years later, of the paper [52] by Hone, Kuznetsov, and Ragnisco, where the Bäcklund transformations for the rational Gaudin magnet have been constructed, and answers to a conjecture of Sklyanin [110] about the extensions of the rational results to the trigonometric and elliptic cases. As it will be clear the trigonometric case has paved the way to get results also for the elliptic case. In the first section I will review the rational results, following [52], [110] and [111], then I will give the results for the trigonometric and elliptic cases.

## 3.1 The rational case

The $\mathfrak{sl}(2)$ rational Gaudin magnet is defined by the Lax matrix:

$$L(\lambda) = \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} s_j^3 & s_j^- \\ s_j^+ & -s_j^3 \end{pmatrix} + \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} A_r(\lambda) & B_r(\lambda) \\ C_r(\lambda) & -A_r(\lambda) \end{pmatrix}, \tag{3.1}$$

$$A_r(\lambda) = \alpha + \sum_{j=1}^{n} \frac{s_j^3}{\lambda - \lambda_j}, \qquad B_r(\lambda) = \sum_{j=1}^{n} \frac{s_j^-}{\lambda - \lambda_j}, \qquad C_r(\lambda) = \sum_{j=1}^{n} \frac{s_j^+}{\lambda - \lambda_j}. \tag{3.2}$$

The dynamical variables of the model are the $3N$ generators of the direct sum of $N$ spins: $s_j^3$, $s_j^\pm$, $j = 1, \ldots, N$. Notice that this Lax matrix differs from (2.20) in the rational case because of the extra term proportional to the $\sigma_3$ matrix. Actually, this is not an essential difference: as I will explain in the next few lines, adding the constant $\alpha$ is the simplest way to recover the $N^{\text{th}}$ integral of the motion of the system, given by the total spin in the $z$ direction, that is $\sum_j s_j^3$. This extra term does not alter the linear $r$-matrix structure (2.21) because it gives rise to a matrix that commutes with $r$, so:

$$\{L_1(\lambda), L_2(\mu)\} = [r(\lambda-\mu), L_1(\lambda) + L_2(\mu)], \tag{3.3}$$

where I recall that in the rational case the $r$-matrix is explicitely given by:

$$r(\lambda) = \frac{i}{\lambda}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.4}$$

Equivalently, the Poisson brackets for the functions $A_r(\lambda)$, $B_r(\lambda)$ and $C_r(\lambda)$ are defined by (see also (1.52)):

$$\{A_r(\lambda), A_r(\mu)\} = \{B_r(\lambda), B_r(\mu)\} = \{C_r(\lambda), C_r(\mu)\} = 0, \tag{3.5}$$

$$\{A_r(\lambda), B_r(\mu)\} = i\frac{B_r(\mu) - B_r(\lambda)}{\lambda - \mu}, \tag{3.6}$$

$$\{A_r(\lambda), C_r(\mu)\} = i\frac{C_r(\lambda) - C_r(\mu)}{\lambda - \mu}, \tag{3.7}$$

$$\{B_r(\lambda), C_r(\mu)\} = i\frac{2(A_r(\mu) - A_r(\lambda))}{\lambda - \mu}, \tag{3.8}$$

and, in turn, these are equivalent to the Poisson brackets

$$\{s_j^3, s_k^\pm\} = \mp \mathrm{i}\delta_{jk} s_k^\pm, \qquad \{s_j^+, s_k^-\} = -2\mathrm{i}\delta_{jk} s_k^3 \tag{3.9}$$

for the $N$ $\mathfrak{sl}(2)$ spins. These brackets have $N$ Casimirs, given by the "length" $s_j^2$ of the spins:

$$s_j^2 = \left(s_j^3\right)^2 + s_j^+ s_j^- . \tag{3.10}$$

The determinant of the Lax matrix is the generating function of the integrals of motion:

$$-\det(L) \doteq \gamma^2(\lambda) = A_r^2(\lambda) + B_r(\lambda)C_r(\lambda) = \alpha^2 + \sum_{j=1}^N \left(\frac{H_j}{\lambda - \lambda_j} + \frac{s_j^2}{(\lambda-\lambda_j)^2}\right), \tag{3.11}$$

where the $N$ Hamiltonians $H_j$ are given by:

$$H_j = \sum_{k\neq j} \frac{2s_j^3 s_k^3 + s_j^+ s_k^- + s_j^- s_k^+}{\lambda_j - \lambda_k} + 2\alpha s_j^3. \tag{3.12}$$

The existence of the $r$-matrix readily implies that these integrals are in involution:

$$\{H_j, H_k\} = 0, \qquad j,k = 1,\ldots,N. \tag{3.13}$$

Note that, as it must be, the total spin in the $z$ direction is conserved thanks to the relation

$$\sum_{j=1}^{N} H_j = 2\alpha J_3, \qquad J_3 \doteq \sum_{j=1}^{N} s_j^3.$$

When $\alpha = 0$, the generating function (3.11) gives $N-1$ independent integrals because $\sum_{j=1}^{N} H_j = 0$ and one can add by hand the integral $J_3$. For future convenience it is also useful to define the other two components of the total spin $J^+$ and $J^-$ (that in the case $\alpha = 0$ are commuting integrals too):

$$J_+ = \sum_{j=1}^{N} s_j^+, \qquad J_- = \sum_{j=1}^{N} s_j^-. \tag{3.14}$$

### 3.1.1 The dressing matrix and the explicit transformations

In [52] the authors made the ansatz of a linear dependence on the spectral parameter $\lambda$ for the dressing matrix $D(\lambda)$. Sklyanin noted [110], [111] that this ansatz coincides with the elementary Lax matrix $L(\lambda)$ for the $XXX$ Heisenberg magnet and showed why it naturally leads to canonical Bäcklund transformation. As I will show, in proving canonicity of the Bäcklund transformations for the trigonometric and elliptic Gaudin magnet, the Sklyanin's arguments will be very useful.
Following [52], let me take the dressing matrix as:

$$D(\lambda) = D_1\lambda + D_0. \tag{3.15}$$

There are some constraints that must be satisfied by the matrices $D_1$ and $D_0$. First of all the matrix $D_1$ has to be diagonal, as it can be easily seen by taking the limit $\lambda \to \infty$ in the equation defining the Bäcklund transformations (1.48):

$$\tilde{L}(\lambda)D(\lambda) = D(\lambda)L(\lambda). \tag{3.16}$$

I recall that by $\tilde{L}$ I denote a Lax matrix having the same spectral dependence as $L$ but with all the dynamical variables $\{s_j^\alpha\}$ substituted by tilded spins $\tilde{s}_j^\alpha$. As explained in the subsection 1.5.2, the zeroes of the determinant of the dressing matrix correspond to the parameters in the Bäcklund transformations. The simplest transformation has only one parameter, so one can require that $\det(D(\lambda)) = 0$ only at one point, say $\lambda = \zeta$. The choices $D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $D_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ give quite similar Bäcklund transformations, the only difference is that they move in opposite direction in discrete time, so one can consider only the first case. The dressing matrix can be then parametrized as follows:

$$D(\lambda) = \begin{pmatrix} \lambda - \zeta + xy & y \\ x & 1 \end{pmatrix}. \tag{3.17}$$

Really in [52] a larger family of dressing matrices is considered, because of an extra parameter $\omega$ entering in $D(\lambda)$:

$$D_\omega(\lambda) = \begin{pmatrix} \lambda - \zeta + xy/\omega & y \\ x & \omega \end{pmatrix}.$$

Here for simplicity I will fix $\omega = 1$. However the dynamical meaning of this parameter that seems to emerge from [52] in a discrete time picture is simple: in fact the Bäcklund transformations generated by the dressing matrix will be canonical transformations with a given generating function. The parameter $\omega$ is the conjugated variable to the integral $J_3$: it will reproduce the discrete time dynamics corresponding to the flow governed by this integral. So far the variables $x$ and $y$ are undetermined dynamical variables (possibly depending on all the components of the spins): note however that comparing the asymptotics in $\lambda$ in both sides of (3.17), they are constrained to be:

$$y = \frac{J_-}{2\alpha}, \qquad x = \frac{\tilde{J}_+}{2\alpha}. \tag{3.18}$$

At this point the similarity transformation produced by the dressing matrix:

$$\tilde{L}(\lambda) = D(\lambda)L(\lambda)D^{-1}(\lambda), \tag{3.19}$$

defines only an *implicit* map, because $x$, and therefore $D(\lambda)$ depends on *tilded* variables. In order to get an *explicit* map one as to take advantage of the spectrality property (see subsection 1.5.2). Since $D(\lambda = \zeta)$ is a rank one matrix, it has a kernel, say $|\Omega\rangle$. The key point is that the kernel depends on $x$. By applying $|\Omega\rangle$ to the equation (3.16), one readily has $D(\zeta)L(\zeta)|\Omega\rangle = 0$, implying that the kernel of the dressing matrix evaluated at $\lambda = \zeta$ is an eigenvector of the Lax matrix evaluated at the same point: $L(\zeta)|\Omega\rangle = \gamma(\zeta)|\Omega\rangle$, where $\gamma(\lambda)$ is defined in (3.11). But this last relation connects the variable $x$ carried by $|\Omega\rangle$ with the untilded dynamical variables. Then the map becomes explicit.

The kernel of $D(\zeta)$ is given by:

$$|\Omega\rangle = \begin{pmatrix} 1 \\ -x \end{pmatrix}. \tag{3.20}$$

The formula:

$$L(\zeta)|\Omega\rangle = \begin{pmatrix} A_r(\zeta) & B_r(\zeta) \\ C_r(\zeta) & -A_r(\zeta) \end{pmatrix} \begin{pmatrix} 1 \\ -x \end{pmatrix} = \gamma(\zeta) \begin{pmatrix} 1 \\ -x \end{pmatrix}$$

implies:

$$x = \frac{A_r(\zeta) - \gamma(\zeta)}{B_r(\zeta)} = -\frac{C_r(\zeta)}{A_r(\zeta) + \gamma(\zeta)}. \tag{3.21}$$

Now it is possible to write down the explicit Bäcklund transformations. Because (3.16) is an equality between rational functions of the spectral parameter, one must equate the residues at the poles $\lambda = \lambda_j$ in order to ensure that this equality holds. So one has:

$$\tilde{L}_j D(\lambda = \lambda_j) = D(\lambda = \lambda_j) L_j, \tag{3.22}$$

where

$$L_j = \begin{pmatrix} s_j^3 & s_j^- \\ s_j^+ & s_j^3 \end{pmatrix}.$$

Explicitly:

$$\tilde{s}_j^3 = \frac{(\lambda_j - \zeta + 2xy)s_j^3 - x(\lambda_j - \zeta + xy)s_j^- + ys_j^+}{(\lambda_j - \zeta)}, \tag{3.23}$$

$$\tilde{s}_j^- = \frac{(\lambda_j - \zeta + xy)^2 s_j^- - 2y(\lambda_j - \zeta + xy)s_j^3 - y^2 s_j^+}{(\lambda_j - \zeta)}, \tag{3.24}$$

$$\tilde{s}_j^+ = \frac{s_j^+ + 2xs_j^3 - x^2 s_j^-}{(\lambda_j - \zeta)}. \tag{3.25}$$

The variables $x$ and $y$ are given by (3.21) and (3.18), which show that they depend on all the dynamical variables of the chain, $\zeta$ being the parameter of the transformations. Note that, by fixing the initial conditions, $\gamma(\zeta)$ becomes a constant, since it is the generating function of the integrals. This implies that the previous are indeed rational transformations.

### 3.1.2 The generating function of the canonical transformations

The maps (3.23), (3.24), (3.25) define canonical transformations from the untilded variables to the tilded ones. This can be showed in various way. Since the maps are explicit, the most direct way is to calculate the Poisson brackets between the tilded variables in order to establish whether the structure (3.9) is retained. But this is the crudest way to proceed. I will follow two finer arguments, the first is the one used in [52] where the authors found the explicit generating function for the one parameter maps (3.23), (3.24), (3.25). The second is due to Sklyanin [111] and is based on the equivalence between the dressing matrix arising from the composition of two Bäcklund transformations with two different parameters (a two-points map) and the one-site Lax matrix for the classical $XXX$ Heisenberg spin chain on the lattice. I will resume the Sklyanin's proof later in this section, when dealing with the two-points map.
The first thing to note is that the transformations (3.23), (3.24), (3.25) are not all independent due to the Casimirs invariance:

$$s_j^2 = \left(s_j^3\right)^2 + s_j^+ s_j^- = \left(\tilde{s}_j^3\right)^2 + \tilde{s}_j^+ \tilde{s}_j^-, \tag{3.26}$$

so that for example one can consider only the transformations $(s_j^3, s_j^-) \mapsto (\tilde{s}_j^3, \tilde{s}_j^+)$, the other $2N$ variables, $s_j^+$ and $\tilde{s}_j^-$, $j = 1..N$ being given by:

$$s_j^+ = \frac{s_j^2 - \left(s_j^3\right)^2}{s_j^-} \qquad \tilde{s}_j^- = \frac{s_j^2 - \left(\tilde{s}_j^3\right)^2}{\tilde{s}_j^+}. \tag{3.27}$$

Then, the generating function $F$ will depends on a set of $2N$ dynamical variables, say $F = F(\tilde{s}_j^+, s_j^-)$, and is such that:

$$s_j^3 = i s_j^- \frac{\partial F(\tilde{s}_j^+, s_j^-)}{\partial s_j^-}, \qquad \tilde{s}_j^3 = i \tilde{s}_j^+ \frac{\partial F(\tilde{s}_j^+, s_j^-)}{\partial \tilde{s}_j^+}, \tag{3.28}$$

In [52] the authors were able to show that the Bäcklund transformations (3.23), (3.24), (3.25) can be rewritten as a map $(s_j^3, \tilde{s}_j^3) \mapsto (s_j^-, \tilde{s}_j^+)$ as follows:

$$s_j^3 = \frac{\tilde{J}_+}{2\alpha}\, s_j^- + z_j, \qquad \tilde{s}_j^3 = \frac{J_-}{2\alpha}\, \tilde{s}_j^+ + z_j, \tag{3.29}$$

where

$$z_j^2 \doteq s_j^2 - (\lambda_j - \zeta)\, \tilde{s}_j^+ s_j^-, \qquad j = 1, \ldots, N. \tag{3.30}$$

By integration one gets the generating function of the canonical transformations [52]:

$$F(\tilde{s}_j^+, s_j^-) = -i \frac{\tilde{J}_+ J_-}{2\alpha} - i \sum_{j=1}^{N} \left( 2z_j + s_j \log \frac{z_j - s_j}{z_j + s_j} \right). \tag{3.31}$$

Now, with the help of equation (3.21), rewritten as $\gamma(\zeta) = A_r(\zeta) - \frac{\tilde{J}_+}{2\alpha} B_r(\zeta)$, it is possible to check that indeed $\gamma(\zeta)$ and $\zeta$ are conjugated variables:

$$\gamma(\zeta) = \frac{\partial F(\tilde{s}_j^+, s_j^-)}{\partial \zeta}.$$

This is the spectrality property.

### 3.1.3 The two points map

Once that a one-parameter Bäcklund transformation is constructed, it is possible to obtain a chain of multi-parametric Bäcklund transformations by an iterative composition:

$$L \mapsto \tilde{L} = D_a L D_a^{-1} \mapsto \tilde{\tilde{L}} = D_b \tilde{L} D_b^{-1} = D_b D_a L \left(D_b D_a\right)^{-1} \mapsto \tilde{\tilde{\tilde{L}}} = \ldots,$$

where $a, b, \ldots$ denote the parameters of every single Bäcklund transformation. The Sklyanin's two-points dressing matrix [110], given by the elementary Lax matrix of the

$XXX$ Heisenberg spin chain on the lattice, is recovered by looking at the transformations generated by:

$$L(\lambda) \mapsto \tilde{\tilde{L}}(\lambda) = \left(D_{\zeta_2}^{-1}(\lambda)D_{\zeta_1}(\lambda)\right) L(\lambda) \left(D_{\zeta_2}^{-1}(\lambda)D_{\zeta_1}(\lambda)\right)^{-1} \doteq D_{\zeta_1,\zeta_2}(\lambda)L(\lambda)D_{\zeta_1,\zeta_2}^{-1}(\lambda), \tag{3.32}$$

where $D_\zeta(\lambda)$ is given by (3.17). The total $D_{\zeta_1,\zeta_2}(\lambda)$ dressing matrix in (3.32) is the composition of the following two elementary dressing matrices:

$$D_{\zeta_1}(\lambda) = \begin{pmatrix} \lambda - \zeta_1 + xy & y \\ x & 1 \end{pmatrix} \qquad D_{\zeta_2}^{-1}(\lambda) = \frac{1}{\lambda - \zeta_2} \begin{pmatrix} 1 & -Y \\ -X & \lambda - \zeta_2 + XY \end{pmatrix}. \tag{3.33}$$

By the asymptotic in $\lambda$ in the two equations $\tilde{L}(\lambda)D_{\zeta_1}(\lambda) = D_{\zeta_1}(\lambda)L(\lambda)$ and $\tilde{\tilde{L}}(\lambda)D_{\zeta_2}(\lambda) = D_{\zeta_2}(\lambda)L(\lambda)$ it is readily seen that $x = X = \frac{\tilde{J}_+}{2\alpha}$, so that in the following I will write $x$ for $X$. The complete dressing matrix is explicitly given by:

$$D_{\zeta_1,\zeta_2}(\lambda) = \begin{pmatrix} \lambda - \zeta_1 + x(y - Y) & y - Y \\ x\left((\zeta_1 - \zeta_2) - x(y - Y)\right) & \lambda - \zeta_2 - x(y - Y) \end{pmatrix}. \tag{3.34}$$

Note that, since (3.32) is an homogeneous equation, I have omitted the multiplicative factor $(\lambda - \zeta_2)^{-1}$ that arises from the inversion of the matrix $D_{\zeta_2}(\lambda)$: this doesn't affect the Bäcklund transformations. Now a change of parameters entering in (3.34) allows to identify this dressing matrix with the one given by Sklyanin in [110], [111]. I pose:

$$\begin{cases} \zeta_1 = \lambda_0 + \mu \\ \zeta_2 = \lambda_0 - \mu \end{cases}, \qquad \begin{cases} x = q \\ y - Y = p \end{cases}.$$

These give the dressing matrix:

$$D_{\lambda_0,\mu}(\lambda) = \begin{pmatrix} \lambda - \lambda_0 - \mu + qp & p \\ q\,(2\mu - qp) & \lambda - \lambda_0 + \mu - qp \end{pmatrix}. \tag{3.35}$$

The symbols used by Sklyanin are for purpose: indeed by a direct calculation it is simple to show that, if $p$ and $q$ are canonical, then (3.35) coincides with the Lax matrix for the elementary $XXX$ Heisenberg spin chain on the lattice having the *quadratic* Poisson brackets:

$$\{D_1(\lambda), D_2(\lambda)\} = [r(\lambda - \mu), L_1(\lambda) \otimes L_2(\mu)],$$

where $r(\lambda)$ is given in 3.4. Obviously this does not mean that in the matrix (3.35) the variables $p$ and $q$ are canonical when one allows them to depend on the dynamical variables of the Gaudin system through the spectrality property. Nevertheless Sklyanin was able to use the resemblance (or even the equivalence if $p$ and $q$ are canonical) of (3.35) with the Lax matrix for the Heisenberg magnet to show the canonicity of the

Bäcklund transformations originating from this matrix: also, if this property is obvious because (3.35) is the composition of two canonical transformations, that therefore must be again canonical, the arguments will be useful in proving the canonicity of the Bäcklund transformations in the trigonometric and elliptic cases, so I now give an account of what was showed in [111].
Consider an extended phase space, given by the dynamical variables of a Lax matrix $L$ and an independent set of variables appearing in the dressing matrix $D$. To fix the ideas, I will call $(X, Y)$ the set of variables appearing in $L$ and $(Q, P)$ the set of variables appearing in $D$. Since the two sets are independent, they Poisson commute. Following [111], without loss of generality I will assume that $(X, Y)$ and $(Q, P)$ are canonical variables. The hypothesis is that both $L$ and $D$ satisfy the same *quadratic* $r$-matrix structure. As a consequence of this and of the Poisson commutativity of $L$ and $D$ it simply follows that also $LD$ and $DL$ satisfy the same $r$-matrix structure as well. Consider now the relation:

$$\tilde{L}(\lambda)\tilde{D}(\lambda) = D(\lambda)L(\lambda). \tag{3.36}$$

This defines a canonical transformation $(X, Y, Q, P) \mapsto (\tilde{X}, \tilde{Y}, \tilde{Q}, \tilde{P})$ since, as we have just seen, $LD$ and $DL$ have the same Poisson structure. So there will be a generating function, say $F(Y, \tilde{Y}, P, \tilde{P})$ such that:

$$X = \frac{\partial F}{\partial Y}\,, \quad \tilde{X} = -\frac{\partial F}{\partial \tilde{Y}}\,, \quad Q = \frac{\partial F}{\partial P}\,, \quad \tilde{Q} = -\frac{\partial F}{\partial \tilde{P}}\,. \tag{3.37}$$

Note that, thanks to the previous generating function one has:

$$\begin{cases} X = X(Y, \tilde{Y}, P, \tilde{P}) \\ \tilde{X} = \tilde{X}(Y, \tilde{Y}, P, \tilde{P}) \\ Q = Q(Y, \tilde{Y}, P, \tilde{P}) \\ \tilde{Q} = \tilde{Q}(Y, \tilde{Y}, P, \tilde{P}). \end{cases}$$

Impose the constraints $\tilde{P} = P$ and $\tilde{Q} = Q$ and use them to express $P$ (and then $\tilde{P}$) as a function of $Y$ and $\tilde{Y}$. This gives also $X$ and $\tilde{X}$ as functions of $Y$ and $\tilde{Y}$. Now, as observed by Sklyanin, the transformation $(X, Y) \to (\tilde{X}, \tilde{Y})$ is a canonical transformation, given by the generating function

$$\mathfrak{F}(Y, \tilde{Y}) = F(Y, \tilde{Y}, P(Y, \tilde{Y}), P(Y, \tilde{Y})),$$

so that:

$$X = \frac{\partial \mathfrak{F}}{\partial Y}, \qquad \tilde{X} = -\frac{\partial \mathfrak{F}}{\partial \tilde{Y}}. \tag{3.38}$$

In fact:

$$\frac{\partial \mathfrak{F}}{\partial Y}=\left.\frac{\partial F}{\partial Y}\right|_{\substack{\tilde{P}=P\\ \tilde{Q}=Q}}+\left.\frac{\partial F}{\partial P}\right|_{\substack{\tilde{P}=P\\ \tilde{Q}=Q}}\frac{\partial P}{\partial Y}+\left.\frac{\partial F}{\partial \tilde{P}}\right|_{\substack{\tilde{P}=P\\ \tilde{Q}=Q}}\frac{\partial \tilde{P}}{\partial Y}.$$

But thanks to (3.37) when $\tilde{P}=P$ and $\tilde{Q}=Q$ the last two terms in this sum vanish, so that

$$\frac{\partial \mathfrak{F}}{\partial Y}=\left.\frac{\partial F}{\partial Y}\right|_{\substack{\tilde{P}=P\\ \tilde{Q}=Q}}=\left.X\right|_{\substack{\tilde{P}=P\\ \tilde{Q}=Q}}.$$

Similarly it is possible to prove that:

$$-\frac{\partial \mathfrak{F}}{\partial \tilde{Y}}=\left.\tilde{X}\right|_{\substack{\tilde{P}=P\\ \tilde{Q}=Q}}.$$

Notice that the dependencies of the functions $P$ and $Q$ on the dynamical variables appearing in the Lax matrix after the constraints $\tilde{P}=P$ and $\tilde{Q}=Q$ have been imposed, must be, by consistency, the same obtained through the spectrality: in fact $\tilde{P}=P$ and $\tilde{Q}=Q$ means no other than $\tilde{D}(\lambda)=D(\lambda)$ or, equivalently, that the relation (3.36) turns into the usual similarity transformation for the Bäcklund transformations, i.e. $\tilde{L}D=DL$. At this point Sklyanin noted [109] how the canonicity of Bäcklund transformations for the rational Gaudin magnet generated by the dressing matrix (3.35) follows as a corollary taking the limit of linear Poisson brackets, as explained in 2.3 or at the end of the appendix B.
What is important for us and has to be fixed in mind is that all this explains why, taking as dressing matrix the one-site Lax matrix of the classical Heisenberg spin chain, one obtains canonical transformations preserving the algebraic form of the integrals of motion (remember that $\tilde{L}$ and $L$ are linked by a similarity transformation); note also how the rational, trigonometric or elliptic dependence by the spectral parameter of the Gaudin model does not enter explicitly in the proof, but only implicitly in the choice of the corresponding dependence of the dressing matrix: the arguments given in the rational case can be repeated in the other two cases.
The explicit Bäcklund transformations corresponding to matrix (3.35) can be obtained by the spectrality property. Now there are two kernels, say $|\Omega_+\rangle$ and $|\Omega_-\rangle$, corresponding respectively to the values $\lambda=\lambda_0+\mu$ and $\lambda=\lambda_0-\mu$:

$$|\Omega_+\rangle=\begin{pmatrix}1\\ -q\end{pmatrix}\qquad |\Omega_-\rangle=\begin{pmatrix}p\\ 2\mu-pq\end{pmatrix}.$$

The eigenvectors relations $L(\lambda=\lambda_0\pm\mu)|\Omega_\pm\rangle=\gamma(\lambda_0\pm\mu)|\Omega_\pm\rangle$, where the function $\gamma(\lambda)$ is defined in (3.11), give the following explicit dependence of the functions $p$ and $q$ in terms of the only "untilded" variables:

$$q=q(\lambda_0+\mu)=\left.\frac{A_r(\lambda)-\gamma(\lambda)}{B_r(\lambda)}\right|_{\lambda=\lambda_0+\mu},\qquad \frac{1}{p}=\frac{q(\lambda_0+\mu)-q(\lambda_0-\mu)}{2\mu}. \tag{3.39}$$

Again the residues at the poles $\lambda = \lambda_j$ must be equated as in (3.22). The formulae obtained read as:

$$\tilde{s}_j^3 = \frac{\left(v_j^2 - \mu^2 + 2pq(2\mu - pq)\right) s_j^3 + p\,(v_j + \mu - pq)\, s_j^+ - q(2\mu - pq)(v_j - \mu + pq)s_j^-}{v_j^2 - \mu^2}\,, \tag{3.40}$$

$$\tilde{s}_j^- = \frac{(v_j - \mu + pq)^2 s_j^- - p^2 s_j^+ - 2p(v_j - \mu + pq)s_j^3}{v_j^2 - \mu^2}\,, \tag{3.41}$$

$$\tilde{s}_j^+ = \frac{(v_j + \mu - pq)^2 s_j^+ - q^2(2\mu - pq)^2 s_j^- + 2q(2\mu - pq)(v_j + \mu - pq)s_j^3}{v_j^2 - \mu^2}\,, \tag{3.42}$$

where for simplicity I posed $v_j = \lambda_j - \lambda_0$. Again, as expected (see at the end of 1.5.2), by fixing the initial conditions, the transformation are rational functions since the function $\gamma$ that enters in $p$ and $q$, defined in (3.11), is the only possible source of irrational terms but is a constant on every orbit.

### 3.1.4 Physical Bäcklund transformations

Suppose to have a real solution of the equations of motion given by some Hamiltonians $\mathfrak{H}$ combination of the Hamiltonians $H_i$ in (3.12); with real solution I mean $3N$ real functions of time $(s_i^1, s_i^2, s_i^3)$, with $s_i^1 \doteq \frac{s_i^+ + s_i^-}{2}$, $s_i^2 \doteq \frac{s_i^+ - s_i^-}{2i}$, solving the system $\dot{s}_i^a = \{s_i^a, \mathfrak{H}\}$, $a = 1, 2, 3$, $i = 1 \ldots N$. The new solutions $(\tilde{s}_i^1 = \frac{\tilde{s}_i^+ + \tilde{s}_i^-}{2}, \tilde{s}_i^2 = \frac{\tilde{s}_i^+ - \tilde{s}_i^-}{2i}, \tilde{s}_i^3)$ given by (3.40), (3.41) and (3.42), in general will be complex because the parameters $\lambda_0$ and $\mu$ are complex. With some constraint on this two parameters, it is however still possible to have a two-parametric family of Bäcklund transformations sending real variables to real variables (with the two parameters real). Bäcklund transformation are obtained by the equivalence (3.22):

$$\tilde{L}_k D_{\lambda_0,\mu}(\lambda = \lambda_k) = D_{\lambda_0,\mu}(\lambda = \lambda_k) L_k, \qquad k = 1 \ldots N. \tag{3.43}$$

If $(s_k^1, s_k^2, s_k^3)$ are $3N$ real triples, then the $N$ matrices $L_k$ are Hermitian. I'm requiring that also $\tilde{L}_k$ have to be Hermitian, and this means that the matrices $D_{\lambda_0,\mu}(\lambda = \lambda_k)$ must be proportional to unitary matrices. I claim that this is the case if $\lambda_0$ is real and $\mu$ is a purely imaginary number. So in the following of the section I put:

$$\mu = i\epsilon, \qquad (\lambda_0, \epsilon) \in \mathbb{R}^2.$$

If $D_{\lambda_0,\mu}(\lambda = \lambda_k)$ are proportional to unitary matrices, then they must have the general form:

$$D_{\lambda_0,\mu}(\lambda = \lambda_k) = \begin{pmatrix} \alpha_k & \beta_k \\ -\bar{\beta}_k & \bar{\alpha}_k \end{pmatrix}, \tag{3.44}$$

where the bar means complex conjugation. I consider the $N$ arbitrary parameters of the model $\lambda_k$ real. First observe that the functions $A_r(\lambda), B_r(\lambda)$ and $C_r(\lambda)$, defined in (3.2), satisfies the equivalences:

$$A_r(\lambda_0+i\epsilon)=\bar{A}_r(\lambda_0-i\epsilon), \qquad B_r(\lambda_0+i\epsilon)=\bar{C}_r(\lambda_0-i\epsilon), \qquad C_r(\lambda_0+i\epsilon)=\bar{B}_r(\lambda_0-i\epsilon). \tag{3.45}$$

By these relations readily follows $\gamma^2(\lambda_0+i\epsilon)=\bar{\gamma}^2(\lambda_0-i\epsilon)$. Remeber that the matrices $D_j$ are written in terms of $p$ and $q$, that are defined by the relations

$$q=q(\lambda_0+i\epsilon)=\frac{A_r(\lambda_0+i\epsilon)-\gamma(\lambda_0+i\epsilon)}{B_r(\lambda_0+i\epsilon)}, \qquad p=\frac{2i\epsilon}{q(\lambda_0+i\epsilon)-q(\lambda_0-i\epsilon)}. \tag{3.46}$$

Now, by specifying the sign of the functions $\gamma$ on the Riemann surface by $\gamma(\lambda_0+i\epsilon)=-\bar{\gamma}(\lambda_0-i\epsilon)$, one has:

$$\bar{q}(\lambda_0+i\epsilon)=-\frac{1}{q(\lambda_0-i\epsilon)}.$$

At this point the identification of the functions $\alpha_k$ and $\beta_k$ in (3.44) is straightforward:

$$\begin{cases} \alpha_k=\lambda_k-\lambda_0+i\epsilon\dfrac{|q|^2-1}{|q|^2+1}, \\ \beta_k=2i\epsilon\dfrac{\bar{q}}{|q|^2+1}. \end{cases} \tag{3.47}$$

Under the assumption made, the matrices $D_k$ are unitary.

### 3.1.5 Interpolating Hamiltonian flow

The maps (3.40), (3.41) and (3.42), in the "physical" realization just described, can be seen as an exact time discretizations of each continuous flow corresponding to the Hamiltonian $H_j$ given in (3.12), with $\epsilon$ playing the role of the discrete time. The real arbitrary parameter $\lambda_0$ can be chosen in such a way to extract, from the Bäcklund transformations, the required discrete dynamic whose continuous interpolating flow is exactly the one governed by the Hamiltonian $H_i$. I will give now a proof of this statement with some little modifications with respect to that appeared in [52] in order to enlighten the direct links between this and the similar constructions of the trigonometric and elliptic cases. To clarify this point, let me take the limit $\epsilon\to 0$ and look at what happens at first order. Take first the limit in the $p$ and $q$ functions; it is easy to obtain the following formulae:

$$q=\frac{A_r(\lambda_0)-\gamma(\lambda_0)}{B_r(\lambda_0)}+O(\epsilon), \tag{3.48}$$

$$p=-i\epsilon\frac{B_r(\lambda_0)}{\gamma(\lambda_0)}+O(\epsilon^2). \tag{3.49}$$

Insert these in the dressing matrix:

$$D(\lambda) = (\lambda - \lambda_0)\mathbb{1} + \\ -\frac{i\epsilon}{\gamma(\lambda_0)} \begin{pmatrix} A_r(\lambda_0) & B_r(\lambda_0) \\ C_r(\lambda_0) & -A_r(\lambda_0) \end{pmatrix} + O(\epsilon^2). \tag{3.50}$$

Now, by taking the limit $\epsilon \to 0$ in the equation defining the Bäcklund transformations $\tilde{L}D = DL$, one obtains the Lax equation for a continuous flow:

$$\dot{L}(\lambda) = [L(\lambda), M(\lambda, \lambda_0)], \tag{3.51}$$

where I have defined the time derivative as:

$$\dot{L} = \lim_{\epsilon \to 0} \frac{\tilde{L} - L}{2\epsilon}, \tag{3.52}$$

so that the matrix $M(\lambda, \lambda_0)$ has the explicit form:

$$\frac{i}{2\gamma(\lambda_0)(\lambda - \lambda_0)} \begin{pmatrix} A_r(\lambda_0) & B_r(\lambda_0) \\ C_r(\lambda_0) & -A_r(\lambda_0). \end{pmatrix} \tag{3.53}$$

Using the Poisson brackets for the functions $A_r(\lambda)$, $B_r(\lambda)$ and $C_r(\lambda)$ given in (3.5), it is not difficult to show that indeed the equations of motion (3.51) are equivalent, by an Hamiltonian point of view, to the equations:

$$\dot{L}(\lambda) = \{\mathcal{H}(\lambda_0), L(\lambda)\}, \tag{3.54}$$

with the Hamilton's function given by:

$$\mathcal{H}(\lambda_0) = \gamma(\lambda_0) = \sqrt{A_r^2(\lambda_0) + B_r(\lambda_0)C_r(\lambda_0)}, \tag{3.55}$$

Remember that $\gamma(\lambda_0)$ contains all the integrals of the system. Furthermore $\lambda_0$ is, so far, an arbitrary real parameter. Let me choose this parameter equal to any of the poles ($\lambda_i$) of the Lax matrix, $\lambda_0 = \delta + \lambda_i$ and take the limit $\delta \to 0$. The Hamilton's function (3.55) goes into:

$$\gamma(\lambda_i + \delta) = \frac{s_i}{\delta} + \frac{H_i}{s_i} + O(\delta), \tag{3.56}$$

so that the corresponding equations of motion become:

$$\dot{L}(\lambda) = \frac{1}{s_i}\{H_i, L(\lambda)\}. \tag{3.57}$$

This shows how the Bäcklund transformations are indeed a set of $N$ discretizing maps, each corresponding to a particular interpolating Hamiltonian $H_i$ of the original continuous system: each map is defined by the relations (3.43), (3.47), with $q$ given by:

$$q = \frac{A_r(\lambda_i + i\epsilon) - \gamma(\lambda_i + i\epsilon)}{B_r(\lambda_i + i\epsilon)}$$

## 3.2 The trigonometric case

The trigonometric case has been the fundamental model that led me to a deeper understanding of the meanings and potentialities of Bäcklund transformations. Furthermore it has smooth the way to the elliptic case. Let me begin by remembering the main features of the model. The Lax matrix is given by the expression:

$$L(\lambda) = \begin{pmatrix} A_t(\lambda) & B_t(\lambda) \\ C_t(\lambda) & -A_t(\lambda) \end{pmatrix}, \tag{3.58}$$

where $A_t(\lambda)$, $B_t(\lambda)$ and $C_t(\lambda)$ are the following trigonometric functions of the spectral parameter $(\lambda)$:

$$A_t(\lambda) = \sum_{j=1}^{N} \cot(\lambda - \lambda_j) s_j^3, \qquad B_t(\lambda) = \sum_{j=1}^{N} \frac{s_j^-}{\sin(\lambda - \lambda_j)}, \qquad C_t(\lambda) = \sum_{j=1}^{N} \frac{s_j^+}{\sin(\lambda - \lambda_j)}. \tag{3.59}$$

I assume by now that $\lambda_j$ are arbitrary but real parameters. The $3N$ dynamical variables $\left(s_j^+, s_j^-, s_j^3\right)$, $j = 1, \ldots, N$, obviously obey to $\oplus^N sl(2)$ algebra, i.e.

$$\left\{s_j^3, s_k^{\pm}\right\} = \mp i \delta_{jk} s_k^{\pm}, \qquad \left\{s_j^+, s_k^-\right\} = -2i\delta_{jk} s_k^3. \tag{3.60}$$

By fixing the $N$ Casimirs

$$\left(s_j^3\right)^2 + s_j^+ s_j^- \doteq s_j^2, \tag{3.61}$$

one gets a symplectic manifold given by the direct product of the correspondent $N$ two-spheres. In a complete equivalent way (see appendix B and subsection 2.3) it is possible to describe the Poisson structure of the model in terms of the $r$-matrix formalism. The Lax matrix, containing the dynamical variables, satisfies the *linear* $r$-matrix Poisson algebra:

$$\left\{L(\lambda) \otimes \mathbb{1}, \mathbb{1} \otimes L(\mu)\right\} = \left[r_t(\lambda - \mu), L(\lambda) \otimes \mathbb{1} + \mathbb{1} \otimes L(\mu)\right], \tag{3.62}$$

where $r_t(\lambda)$ stands for the trigonometric $r$ matrix [35]:

$$r_t(\lambda) = \frac{i}{\sin(\lambda)} \begin{pmatrix} \cos(\lambda) & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cos(\lambda) \end{pmatrix}. \tag{3.63}$$

Equations (3.62) in turn are equivalent to the following Poisson brackets for the functions (3.59):

$$\begin{aligned}
\{A_t(\lambda), A_t(\mu)\} = \{B_t(\lambda), B_t(\mu)\} = \{C_t(\lambda), C_t(\mu)\} = 0,\\
\{A_t(\lambda), B_t(\mu)\} = i\frac{\cos(\lambda-\mu)B_t(\mu) - B_t(\lambda)}{\sin(\lambda-\mu)},\\
\{A_t(\lambda), C_t(\mu)\} = i\frac{C_t(\lambda) - \cos(\lambda-\mu)C_t(\mu)}{\sin(\lambda-\mu)},\\
\{B_t(\lambda), C_t(\mu)\} = i\frac{2(A_t(\mu) - A_t(\lambda))}{\sin(\lambda-\mu)},
\end{aligned} \tag{3.64}$$

and these in turn entail (3.60) through (3.59). The determinant of the Lax matrix is again the generating function of the integrals of motion:

$$-\det(L) = A_t^2(\lambda) + B_t(\lambda)C_t(\lambda) = \sum_{i=1}^{N}\left(\frac{s_i^2}{\sin^2(\lambda-\lambda_i)} + H_i\cot(\lambda-\lambda_i)\right) - H_0^2, \tag{3.65}$$

where the $N$ Hamiltonians $H_i$ are of the form:

$$H_i = \sum_{k\neq i}^{N}\frac{2\cos(\lambda_i-\lambda_k)s_i^3 s_k^3 + s_i^+ s_k^- + s_i^- s_k^+}{\sin(\lambda_i-\lambda_k)}. \tag{3.66}$$

Oppositely to the rational case, now only $N-1$ among these Hamiltonians are independent. In fact $\sum_i H_i = 0$. As can be seen by the generating function (3.65), the other integral is given by $H_0$, the projection of the total spin on the $z$ axis (remember also what has been said in subsection 2.2 about the trigonometric case):

$$H_0 = \sum_{j=1}^{N} s_j^3 \doteq J^3 \tag{3.67}$$

By the existence of the $r$-matrix, the involutivity of the Hamiltonians (3.60) follows [9]:

$$\{H_i, H_j\} = 0 \quad i,j = 0,\ldots,N-1. \tag{3.68}$$

The corresponding Hamiltonian flows are governed by the set of equations:

$$\frac{ds_j^3}{dt_i} = \{H_i, s_j^3\}, \qquad \frac{ds_j^\pm}{dt_i} = \{H_i, s_j^\pm\}. \tag{3.69}$$

### 3.2.1 The dressing matrix and the explicit transformations

In planning the construction of Bäcklund transformations for the trigonometric case, O. Ragnisco and I have used two different approaches in the study of the problem and in the search for the dressing matrix. Initially we expected to find two classes of maps for the model; soon after however we realize that indeed the two approaches are perfectly equivalent, leading to the same transformations. For completeness I will report both of them, giving also the "dictionary" to pass from one to another.

**First approach** At the core of the first approach there is the observation that the trigonometric Gaudin model with $N$ sites is just the rational Gaudin model with $2N$ sites with an extra reflection symmetry. In fact, performing the following change of variable:

$$\lambda \to z \doteq e^{i\lambda},$$

the $N$-sites trigonometric Lax matrix takes a rational form in $z$:

$$L(z) = iJ^3\sigma_3 + \sum_{j=1}^{N} \left( \frac{L_1^j}{z - z_j} - \sigma_3 \frac{L_1^j}{z + z_j} \sigma_3 \right), \tag{3.70}$$

where the matrices $L_1^j, \quad j = 1, \ldots, N$, are defined by:

$$L_1^j = iz_j \begin{pmatrix} s_j^3 & s_j^- \\ s_j^+ & -s_j^3 \end{pmatrix}.$$

Equation (3.70) entails the following involution on $L(z)$:

$$L(z) = \sigma_3 L(-z)\sigma_3. \tag{3.71}$$

Obviously also $\tilde{L}$ has to enjoy the same reflection symmetry (3.71). This is automatically verified if the dressing matrix $D(\lambda)$ share with $L$ the same symmetry (3.71). So the elementary choice for the spectral structure of the $D(\lambda)$ has to preserve (3.71) and contains only one pair of opposite singular points. It reads:

$$D = D_\infty + \frac{D_1}{z - \xi} - \frac{\sigma_3 D_1 \sigma_3}{z + \xi}. \tag{3.72}$$

By taking the limit $z \to \infty$ in

$$\tilde{L}(z)D(z) = D(z)L(z), \tag{3.73}$$

it is readily seen that $D_\infty$ has to be a diagonal matrix. For bounded values of $z$, equation (3.73) requires that both sides have equal residues at the simple poles $\pm z_j, \pm\xi$.

However, in view of the symmetry property (3.71), it will be enough to look only at half of them, say $+z_j$ and $+\xi$. The corresponding equations will be:

$$\tilde{L}_1^{(j)} D(z_j) = D(z_j) L_1^{(j)}, \tag{3.74}$$

$$\tilde{L}(\xi) D_1 = D_1 L(\xi). \tag{3.75}$$

In principle, equations (3.74), (3.75) yield a dressing matrix depending *both* on the old (untilded) variables and the new (tilded) ones, implying in turn an implicit relationship between the same variables. As usually, to get an explicit relationship, one has to resort to the spectrality property. In this case it is sufficient to force the determinant of the dressing matrix $D(z)$ to have a pair of opposite *nondynamical* zeroes, say at $z = \pm\eta$, and to allow the matrix $D_1$ to be proportional to a projector. Again by symmetry it suffices to consider just one of the zeroes. With the same arguments for the rational case, if $\eta$ is a zero of $\det D(z)$, then $D(\eta)$ is a rank one matrix, possessing a one dimensional kernel $|K(\eta)\rangle$; the equation $\tilde{L}(z)D(z) = D(z)L(z)$ entails $D(\eta)L(\eta)|K(\eta)\rangle = 0$ that in turn allows to infer that $|K(\eta)\rangle$ is an eigenvector for the Lax matrix $L(\eta)$:

$$L(\eta)|K(\eta)\rangle = \mu(\eta)|K(\eta)\rangle. \tag{3.76}$$

This relation gives a direct link between the parameters appearing in the dressing matrix $D$ and the untilded dynamical variables in $L$. But thanks to (3.75), we have another one dimensional kernel $|K(\xi)\rangle$, because also $D_1$ is a rank 1 matrix:

$$L(\xi)|K(\xi)\rangle = \mu(\xi)|K(\xi)\rangle. \tag{3.77}$$

The two spectrality conditions (3.76), (3.77) are sufficient to fully characterize $D$ in terms of only the old dynamical variables and of the two Bäcklund parameters $\xi$ and $\eta$. The Bäcklund transformations arising from equation (3.74) are then explicit. To clarify this point is better to take a modified point of view on the dressing matrix. First of all note that, requiring that $D_1$ is a rank one matrix, amounts to require that the determinant of $(z^2 - \xi^2)D(z)$ is zero for $z = \xi$ or, by symmetry, for $z = -\xi$. In fact:

$$(z^2 - \xi^2)D(z)|_{z=\xi} = 2\xi D_1, \qquad (z^2 - \xi^2)D(z)|_{z=-\xi} = 2\xi \sigma^3 D_1 \sigma^3. \tag{3.78}$$

Since two dressing matrices differing just by a multiplicative scalar factor define the same BT, one can choose to work with a modified dressing matrix $D'(z)$ defined by the relation:

$$D'(z) \equiv \frac{z^2 - \xi^2}{z} D(z). \tag{3.79}$$

Now the spectrality property requires that the determinant of $D'(z)$ is zero for $z = \xi$ and $z = \eta$. The form that the dressing matrix $D'(z)$ assumes can be further simplified by writing:

$$D'(z) = z^{-1}\hat{A} + \hat{B} + \hat{C}z. \tag{3.80}$$

The matrix $\hat{C}$ is immediately seen to be a diagonal one by looking at the behavior for large values of $z$. On the other hand, $L(0)$ as well as its dressed version $\tilde{L}(0)$ are diagonal matrices:

$$L(0) = iJ^3\sigma_3 - \sum_{j=1}^{N} \frac{L_1^{(j)} + \sigma_3 L_1^{(j)}\sigma_3}{z_j}. \tag{3.81}$$

This readily implies that $\hat{A}$ in (3.80) is diagonal. In turn, (3.71) implies that, if even powers of $z$ are diagonal,odd powers must be off-diagonal, entailing that $\hat{B}$ is an off-diagonal matrix. The two matrices $\hat{A}$ and $\hat{C}$ are then given by $\mathrm{diag}(a_1, a_2)$ and $\mathrm{diag}(c_1, c_2)$, whereas the off-diagonal matrix B is given by $\begin{pmatrix} 0 & b_1 \\ b_2 & 0 \end{pmatrix}$. At this point, thanks to the spectrality property, one can get a deeper insight on the parametrization of matrices $\hat{A}, \hat{B}, \hat{C}$. In fact, if $D'(\xi)$ and $D'(\eta)$ are rank one matrices, there must exist a function of one variable, say $p$, such that the following system is fulfilled:

$$\begin{cases} c_1\xi + a_1/\xi + b_1 p(\xi) = 0 \\ b_2 + p(\xi)(c_2\xi + a_2/\xi) = 0 \end{cases} \tag{3.82}$$

$$\begin{cases} c_1\eta + a_1/\eta + b_2 p(\eta) = 0 \\ b_2 + p(\eta)(c_2\eta + a_2/\eta) = 0. \end{cases} \tag{3.83}$$

The four equations (3.82, 3.83) leave two undetermined parameters, one of which is an inessential global multiplicative factor for $D'(z)$, say $\beta$. The other is indicated with $b$ and will be fixed in the next paragraph in order to show how the matrix $D(z)$ is indeed equivalent to the elementary $XXZ$ classical Heisenberg spin chain on the lattice. Taking into account also (3.79), the total result for $D(z)$ is:

$$D(z) = \frac{\beta z}{z^2 - \xi^2} \begin{pmatrix} \frac{z(p(\eta)\eta - p(\xi)\xi)}{b} + \frac{(p(\xi)\eta - p(\eta)\xi)\eta\xi}{bz} & \frac{\xi^2-\eta^2}{b} \\ \frac{bp(\xi)p(\eta)\left(\xi^2-\eta^2\right)}{\eta\xi} & \frac{b(p(\eta)\eta - p(\xi)\xi)}{z} + \frac{bz(p(\xi)\eta - p(\eta)\xi)}{\eta\xi} \end{pmatrix}. \tag{3.84}$$

The functions $p(\eta)$ and $p(\xi)$ characterize completely the kernels of $D(\eta)$ and $D(\xi)$: in fact they are given by the simple formulae:

$$|K(\xi)\rangle = \begin{pmatrix} 1 \\ p(\xi) \end{pmatrix}, \qquad |K(\eta)\rangle = \begin{pmatrix} 1 \\ p(\eta) \end{pmatrix}, \tag{3.85}$$

and are eigenvectors respectively of $L(\xi)$ and $L(\eta)$, so that by the relations $L(\xi)|K(\xi)\rangle = \mu(\xi)|K(\xi)\rangle$ and $L(\eta)|K(\eta)\rangle = \mu(\eta)|K(\eta)\rangle$ one explicitly has:

$$p(\xi) = \frac{\mu(\xi) - A_t(\xi)}{B_t(\xi)}, \qquad p(\eta) = \frac{\mu(\eta) - A_t(\eta)}{B_t(\eta)} \tag{3.86}$$

with the function $\mu(z)$ defined by the relation $\mu^2(z) = A_t^2(z) + B_t(z)C_t(z)$. Now the matrix (3.84) contains just one set of dynamical variables, so it is possible to give the explicit formulae for the Bäcklund transformations. They can be easily found by the equations for the residues at $\lambda = \lambda_j$ (3.74):

$$\begin{aligned}\tilde{s}_k^3 =& \frac{p(\xi)p(\eta)\left(\xi^2-\eta^2\right)\left(\left({z_k}^2-\eta^2\right)p(\xi)\xi-\left({z_k}^2-\xi^2\right)p(\eta)\eta\right)s_k^- z_k}{\Delta_k}+\\ & \frac{\left(\xi^2-\eta^2\right)\left(\left({z_k}^2-\xi^2\right)p(\xi)\eta-p(\eta)\xi\left({z_k}^2-\eta^2\right)\right)s_k^+ z_k}{\Delta_k}+\\ & \frac{s_k^3\Big[p(\xi)p(\eta)\left(\left(\xi^2+{z_k}^2\right)\left(\eta^2+{z_k}^2\right)-\left(\eta^2+\xi^2\right)-8\eta^2\xi^2{z_k}^2\right)}{\Delta_k}+\\ & -\frac{\left(\eta\xi\left(\xi^2-{z_k}^2\right)\left(\eta^2-{z_k}^2\right)\left(p(\xi)^2+p(\eta)^2\right)\right)\Big]}{\Delta_k},\end{aligned} \tag{3.87a}$$

$$\begin{aligned}\tilde{s}_k^+ =& -\frac{b^2 p(\xi)^2 p(\eta)^2\left(\eta^2-\xi^2\right)^2 s_k^- z_k^2}{\xi\eta\Delta_k}+\frac{b^2\left(\left({z_k}^2-\xi^2\right)p(\xi)\eta-p(\eta)\xi\left({z_k}^2-\eta^2\right)\right)^2 s_k^+}{\eta\xi\Delta_k}+\\ & \frac{2b^2 p(\xi)p(\eta)\left(\xi^2-\eta^2\right)\left(\left({z_k}^2-\xi^2\right)p(\xi)\eta-p(\eta)\xi\left({z_k}^2-\eta^2\right)\right)s_k^3 z_k}{\eta\xi\Delta_k},\end{aligned} \tag{3.87b}$$

$$\begin{aligned}\tilde{s}_k^- =& -\frac{\left(\eta^2-\xi^2\right)^2 s_k^+ z_k^2\xi\eta}{b^2\Delta_k}+\frac{\left(\left({z_k}^2-\eta^2\right)p(\xi)\xi-\left({z_k}^2-\xi^2\right)p(\eta)\eta\right)^2 s_k^-\xi\eta}{b^2\Delta_k}+\\ & \frac{2\left(\xi^2-\eta^2\right)\left(\left({z_k}^2-\eta^2\right)p(\xi)\xi-\left({z_k}^2-\xi^2\right)p(\eta)\eta\right)s_k^3 z_k\xi\eta}{b^2\Delta_k},\end{aligned} \tag{3.87c}$$

where for simplicity I have posed:

$$\Delta_k = (z_k^2-\xi^2)(z_k^2-\eta^2)(p(\xi)\eta-p(\eta)\xi)(p(\eta)\eta-p(\xi)\xi). \tag{3.88}$$

Formulae (3.87a), (3.87b), (3.87c) define a two-parameter Bäcklund transformation: it isn't simple to see that indeed these maps, in this form, are a direct generalizations of the rational ones, (3.40), (3.41) and (3.42). One could expect that the right generalization of the rational dressing matrix has something to do with the classical, partially anisotropic, Heisenberg spin chain on the lattice. Alternatively to the first approach one could have made indeed this ansatz from the very beginning and then analyze if it works. Now, on the contrary, I will show that if one fixes the value of the parameter $b$ in (3.84) as:

$$b = i\sqrt{\eta\xi}, \tag{3.89}$$

then the dressing matrix really goes into the one-site Lax matrix of the Heisenberg spin chain on the lattice. Due to the homogeneity of the relation $\tilde{L}D = DL$, the global multiplicative factor for the dressing matrix are inessential, so I omit the term $\frac{\beta z}{z^2-\xi^2}$ in (3.84). Taking into account (3.89), the diagonal part $D_d$ of (3.84) is:

$$D_d = \frac{i}{2}\Big((p(\xi) - p(\eta))(v - w)\mathbb{1} + (p(\xi) + p(\eta))(v + w)\sigma_3\Big), \tag{3.90}$$

where $v(\xi, \eta)$ and $w(\xi, \eta)$ are given by:

$$v(\xi, \eta) = \frac{z\xi}{\sqrt{\eta\xi}} - \frac{\eta\sqrt{\eta\xi}}{z}, \qquad w(\xi, \eta) = \frac{\xi\sqrt{\eta\xi}}{z} - \frac{z\eta}{\sqrt{\eta\xi}} = -v(\eta, \xi). \tag{3.91}$$

Now I substitute:

$$\xi \to e^{i\zeta_1}, \qquad \eta \to e^{i\zeta_2}, \qquad z \to e^{i\lambda}, \tag{3.92}$$

and take a suitable redefinition of the Bäcklund parameters to clarify the structure of the $D$ matrix:

$$\lambda_0 \doteq \frac{\zeta_1 + \zeta_2}{2}, \qquad \mu \doteq \frac{\zeta_1 - \zeta_2}{2}. \tag{3.93}$$

With these positions it is simple to find that $v - w = 4ie^{i\lambda_0}\sin(\lambda - \lambda_0)\cos(\mu)$ and $v + w = 4ie^{i\lambda_0}\cos(\lambda - \lambda_0)\sin(\mu)$. Considering equation (3.90) jointly with the off-diagonal part of (3.84), the total dressing matrix can be written as:

$$\begin{aligned} D(\lambda) =& \alpha\Big[\sin(\lambda - \lambda_0)\mathbb{1} + \frac{p(\zeta_1) + p(\zeta_2)}{p(\zeta_1) - p(\zeta_2)}\tan(\mu)\cos(\lambda - \lambda_0)\sigma_3 + \\ &+ \frac{2\sin(\mu)}{p(\zeta_2) - p(\zeta_1)}\begin{pmatrix} 0 & 1 \\ -p(\zeta_1)p(\zeta_2) & 0 \end{pmatrix}\Big], \end{aligned} \tag{3.94}$$

where $\alpha$ is the global factor $2e^{i\lambda_0}(p(\zeta_2) - p(\zeta_1))$. Observe that in formula (3.94), with some abuse of notation, $p(\zeta_1)$ and $p(\zeta_2)$ stands of course for $p(\xi)|_{\xi=e^{i\zeta_1}}$ and $p(\eta)|_{\eta=e^{i\zeta_2}}$. Now the most is done. In fact, introducing the two new functions, $P$ and $Q$:

$$p(\zeta_1) = -Q, \qquad p(\zeta_2) = \frac{2\sin(\mu)}{P} - Q. \tag{3.95}$$

the equation (3.94) becomes:

$$D(\lambda) = \alpha\begin{pmatrix} \sin(\lambda - \lambda_0 - \mu) + PQ\cos(\lambda - \lambda_0) & P\cos(\mu) \\ Q\sin(2\mu) - PQ^2\cos(\mu) & \sin(\lambda - \lambda_0 + \mu) - PQ\cos(\lambda - \lambda_0) \end{pmatrix}. \tag{3.96}$$

This is the dressing matrix for the trigonometric Gaudin magnet that generalizes in the more direct way the rational dressing matrix (3.35): also all the remarkable features of the Bäcklund transformations obtained in the previous section can be generalized to the trigonometric case. Furthermore the rational maps are recovered in the limit of "small angles". In order to find the explicit transformations for the dressing matrix (3.96), it is possible to repeat all the arguments about spectrality. Indeed now, in parallel with the rational case, $D(\lambda = \lambda_0 + \mu)$ and $D(\lambda = \lambda_0 - \mu)$ are rank one matrices. So if $|\Omega_+\rangle$ and $|\Omega_-\rangle$ are their respective kernels one has that $|\Omega_+\rangle$ and $|\Omega_-\rangle$ are also the eigenvectors of $L(\lambda_0 + \mu)$ and $L(\lambda_0 - \mu)$ with eigenvalues $\gamma_+$ and $\gamma_-$ where

$$\gamma_\pm = \gamma(\lambda)|_{\lambda=\lambda_0\pm\mu}\,, \qquad \gamma^2(\lambda) \doteq A_t^2(\lambda) + B_t(\lambda)C_t(\lambda) = -\det(L(\lambda)). \tag{3.97}$$

The two kernels are given by:

$$|\Omega_+\rangle = \begin{pmatrix} 1 \\ -Q \end{pmatrix}, \qquad |\Omega_-\rangle = \begin{pmatrix} P \\ 2\,\sin(\mu) - PQ \end{pmatrix} \tag{3.98}$$

and the eigenvectors relations yields the following expression of $P$ and $Q$ in terms of the old variables only:

$$Q = Q(\lambda_0 + \mu) = \left.\frac{A_t(\lambda) - \gamma(\lambda)}{B_t(\lambda)}\right|_{\lambda=\lambda_0+\mu}, \qquad \frac{1}{P} = \frac{Q(\lambda_0 + \mu) - Q(\lambda_0 - \mu)}{2\,\sin(\mu)}. \tag{3.99}$$

As for the rational case, the explicit maps can be found by equating the residues at the poles $\lambda = \lambda_k$ in (3.73), that is by the relation:

$$\tilde{L}_k D_k = D_k L_k, \tag{3.100}$$

where

$$L_k = \begin{pmatrix} s_k^3 & s_k^- \\ s_k^+ & -s_k^3 \end{pmatrix}, \qquad D_k = D(\lambda = \lambda_k). \tag{3.101}$$

The maps read:

$$\begin{aligned}\tilde{s}_k^3 =& \frac{2\,\cos^2(\mu) - (\cos^2(\mu) + \cos^2(\delta_0^k))(1 - 2PQ\,\sin(\mu) + P^2Q^2)}{\Gamma_k} s_k^3 + \\ &+ \frac{P\,\cos(\mu)(\sin(\delta_+^k) - PQ\,\cos(\delta_0^k))}{\Gamma_k} s_k^+ + \\ &- \frac{Q\,\cos(\mu)(2\,\sin(\mu) - PQ)(\sin(\delta_-^k) + PQ\,\cos(\delta_0^k))}{\Gamma_k} s_k^-\,,\end{aligned} \tag{3.102a}$$

$$\begin{aligned}\tilde{s}_k^+ =& \frac{(\sin(\delta_+^k) - PQ\,\cos(\delta_0^k))^2}{\Gamma_k} s_k^+ - \frac{(Q^2\cos^2(\mu)(2\,\sin(\mu) - PQ))^2}{\Gamma_k} s_k^- + \\ &+ \frac{2Q\,\cos(\mu)(2\,\sin(\mu) - PQ)(\sin(\delta_+^k) - PQ\,\cos(\delta_0^k))}{\Gamma_k} s_k^3,\end{aligned} \tag{3.102b}$$

$$\begin{aligned}\tilde{s}_k^- =&\frac{(\sin(\delta_-^k)+PQ\,\cos(\delta_0^k))^2}{\Gamma_k}s_k^- - \frac{P^2\,\cos^2(\mu)}{\Gamma_k}s_k^+ + \\ &-\frac{2P\,\cos(\mu)(\sin(\delta_-^k)+PQ\,\cos(\delta_0^k))}{\Gamma_k}s_k^3.\end{aligned} \tag{3.102c}$$

Note that these expressions could be find also performing the needed changes of variables in (3.87a), (3.87b), (3.87c) In the previous, for simplicity I have put:

$$\begin{cases}\delta_0^k = \lambda_k - \lambda_0 \\ \delta_\pm^k = \lambda_k - \lambda_0 \pm \mu\end{cases} \tag{3.103}$$

and $\Gamma_k$ denotes the determinant of $D(\lambda_k)$:

$$\Gamma_k := \sin(\lambda_k - \lambda_0 - \mu)\sin(\lambda_k - \lambda_0 + \mu)(1 - 2PQ\,\sin(\mu) + P^2Q^2).$$

The rational limit corresponds to small angles $\lambda_0$ and $\mu$: as explained in subsection 2.2, by taking $\lambda_0 \to h\lambda_0$, $\mu \to h\mu$ and $\lambda \to h\lambda$, where $h$ is the expansion parameter, one has:

$$\cot(\lambda-\lambda_k) = \frac{1}{h(\lambda-\lambda_k)} + O(h), \qquad \frac{1}{\sin(\lambda-\lambda_k)} = \frac{1}{h(\lambda-\lambda_k)} + O(h),$$

so that $Q = q^r + O(h^2)$, where the superscript $r$ stands for "rational": $q^r$ coincides with the variable $q$ given in (3.39). The expansion of the variable $P$ gives:

$$P = h(p^r + O(h^2)), \qquad \text{where} \qquad p^r = \frac{2\mu}{q^r(\lambda_0+\mu) - q^r(\lambda_0-\mu)},$$

so that also $p^r$ coincides with the function $p$ in (3.39). Inserting these expressions in the dressing matrix (3.96), one easily finds:

$$D(\lambda) = hD^r(\lambda) + O(h^3), \tag{3.104}$$

where

$$D^r(\lambda) = \begin{pmatrix} \lambda - \lambda_0 - \mu + p^r q^r & p^r \\ q^r(2\mu - p^r q^r) & \lambda - \lambda_0 + \mu - p^r q^r \end{pmatrix}. \tag{3.105}$$

This is exactly the matrix (3.35); in the limit of small angles, the maps (3.102a), (3.102b), (3.102c) obviously reduce to the rational maps (3.40), (3.41), (3.42).

### 3.2.2 Canonicity

The above correspondence with the rational Bäcklund transformations in the limit of small angles shows that the transformations are surely canonical in this limit. The question is if they are canonical *in toto*. As the maps are explicit, one could check by brute-force calculations whether the Poisson structure (3.60) is preserved by tilded variables. However I suggested (see subsection 3.1.3) that the argumens followed by Sklyanin [111], reported in the previous section in the second proof of canonicity of rational Bäcklund transformations, works also in this case if the dressing matrix obeys to the *quadratic* $r$-matrix relation:

$$\{D^1(\lambda), D^2(\tau)\} = [r_t(\lambda-\tau), D^1(\lambda)\otimes D^2(\tau)], \tag{3.106}$$

with $r_t$ the trigonometric dressing matrix (3.63). All the other points in the Sklyanin's proof remain unaltered, so, all that one has to do, is to ascertain that there exist a choice for the Poisson bracket $\{P, Q\}$ such that the relation (3.106) is fulfilled. It is clear in fact that $D(\lambda)$ cannot have this Poisson structure for any Poisson bracket between $P$ and $Q$. As I showed in the previous section, in the rational case the dressing matrix has the quadratic Poisson structure imposed by the rational $r$-matrix provided $p$ and $q$ (the rational functions corresponding to $P$ and $Q$ in (3.96)) are canonically conjugated in the extended space, and this is why they were symbolized by $p$ and $q$; in the trigonometric case, as I'm going to show, $P$ and $Q$ are no longer canonically conjugated but this property must be recovered at order $h$ in the small angle limit. First note that $D(\lambda)$ can be conveniently written as:

$$D(\lambda) = \alpha\,\cos(\mu)\Big[\sin(\lambda)\mathbb{1} + a\,\cos(\lambda)\sigma_3 + \begin{pmatrix} 0 & b \\ c & 0\end{pmatrix}\Big], \tag{3.107}$$

where the coefficients $a, b, c$ are given by:

$$a = \frac{PQ - \sin(\mu)}{\cos(\mu)}, \quad b = P, \quad c = 2Q\,\sin(\mu) - PQ^2. \tag{3.108}$$

Inserting (3.107) in (3.106) one has the following constraints:

$$\{\alpha, \alpha a\} = 0 \quad\Longrightarrow\quad \alpha = \alpha(PQ), \tag{3.109}$$

$$\{\alpha, \alpha b\} = -\alpha^2 ab \quad\Longrightarrow\quad \{\alpha, P\} = \alpha P\frac{\sin(\mu) - PQ}{\cos(\mu)}, \tag{3.110}$$

$$\{\alpha, \alpha c\} = \alpha^2 ac \quad\Longrightarrow\quad \{\alpha, Q\} = -\alpha Q\frac{\sin(\mu) - PQ}{\cos(\mu)}. \tag{3.111}$$

All the remaining relations, namely

$$\{\alpha b, \alpha c\} = 2\alpha^2 a, \quad \{\alpha a, \alpha b\} = \alpha^2 b, \quad \{\alpha a, \alpha c\} = -\alpha^2 c, \tag{3.112}$$

give the same constraint, i.e.:

$$\{Q, P\} = \frac{1 + P^2Q^2 - 2PQ\sin(\mu)}{\cos(\mu)}. \tag{3.113}$$

This expression can be used to find, after a simple integration,

$$\alpha(PQ) = \frac{k}{\sqrt{(1 + P^2Q^2 - 2PQ\sin(\mu))}},$$

so that the Darboux matrix (3.96) is fixed (up to the constant multiplicative factor $k$). Note that the matrix (3.35) has the same dependence in the spectral parameter as the elementary Lax matrix for the classical $xxz$ Heisenberg spin chain on the lattice [35]. Moreover it satisfies also the same quadratic Poisson bracket. This suggests that indeed $D(\lambda)$ can be explicitly recast in the form (see [35]):

$$D(\lambda) = \mathscr{S}_0 1 + \frac{i}{\sin(\lambda)}\big(\mathscr{S}_1\sigma_1 + \mathscr{S}_2\sigma_2 + \cos(\lambda)\mathscr{S}_3\sigma_3\big), \tag{3.114}$$

where the $\sigma_i$'s are the Pauli matrices and the variables $\mathscr{S}_i$ satisfies the Poisson bracket ([35]):

$$\begin{aligned} \{\mathscr{S}_i, \mathscr{S}_0\} &= J_{jk}\mathscr{S}_j\mathscr{S}_k, \\ \{\mathscr{S}_i, \mathscr{S}_j\} &= -\mathscr{S}_0\mathscr{S}_k, \end{aligned} \tag{3.115}$$

where $(i, j, k)$ is a cyclic permutation of $(1, 2, 3)$ and $J_{jk}$ is antisymmetric with $J_{12} = 0, J_{13} = J_{23} = 1$. Indeed it is straightforward to show that the link between the two representations (3.107) and (3.114), up to the factor $\cos(\mu)\sin(\lambda)$ that does not affect neither (3.73) nor the Poisson bracket (3.106), is given by :

$$\alpha = \mathscr{S}_0, \quad -\frac{i\alpha}{2}(b + c) = \mathscr{S}_1, \quad \frac{\alpha}{2}(b - c) = \mathscr{S}_2, \quad -ia\alpha = \mathscr{S}_3, \tag{3.116}$$

and the Poisson brackets (3.109), (3.110), (3.111), (3.112) correspond to those given in (3.115). The dressing matrices in the rational and trigonometric cases are then respectively the Lax matrices of isotropic and partially anisotropic Heisenberg chain: it is not hazardous to take as an ansatz for the dressing matrix of the elliptic Gaudin in the next section the $XYZ$ Heisenberg Lax matrix.

### 3.2.3 Physical Bäcklund transformations

As in the rational case, the trigonometric Bäcklund transformations (3.102a), (3.102b), (3.102c) do not map, in general, real variables into real variables. Again the sufficient condition to ensure this property is to have $\lambda_0$ real and $\mu$ purely imaginary, so again I put hereafter in this section:

$$\mu = i\epsilon, \qquad (\lambda_0, \epsilon) \in \mathbb{R}^2. \tag{3.117}$$

Since the transformations are obtained, as in the rational case, equating the residues at the poles in the equation $\tilde{L}(\lambda)D(\lambda) = D(\lambda)L(\lambda)$, that is by (3.100), the observations made in that case still works: starting from a real solution means to start from an Hermitian $L_k$ in (3.100). Thus, if the transformed matrix $\tilde{L}_k$ has to be Hermitian too, the Darboux matrix must be proportional to an unitary matrix. This is the case if the assumptions (3.117) are verified and, as in the rational case, if $\gamma(\lambda_0+\mu) = -\bar{\gamma}(\lambda_0-\mu)$, where the function $\gamma(\lambda)$ is defined in (3.97)). Note that the condition on the $\gamma$'s specifies their relative sign, the sheet on the Riemann surface, inessential for the spectrality property. The Darboux matrix at $\lambda = \lambda_k$ can be rewritten as:

$$D_k = \begin{pmatrix} \sin(v_k - i\epsilon) + PQ\, \cos(v_k) & P\, \cosh(\epsilon) \\ Q\, \cosh(\epsilon)\,(2i\, \sinh(\epsilon) - PQ) & \sin(v_k + i\epsilon) - PQ\, \cos(v_k) \end{pmatrix}, \tag{3.118}$$

where $v_k \doteq \lambda_k - \lambda_0$. Recall that I'm assuming that the parameters $\lambda_k$ of the model are real. As in the rational case, it is straightforward to show that the relations between the elements of the Lax matrix and their conjugate are:

$$A_t(\lambda_0+i\epsilon) = \bar{A}_t(\lambda_0-i\epsilon); \quad B_t(\lambda_0+i\epsilon) = \bar{C}_t(\lambda_0-i\epsilon); \quad C_t(\lambda_0+i\epsilon) = \bar{B}_t(\lambda_0-i\epsilon). \tag{3.119}$$

If $D_k$ is proportional to a unitary matrix, then the off-diagonal terms of $D_k D_k^\dagger$ have to be zero: this is equivalent to the fulfilling of the equation:

$$P(\sin(v_k-i\epsilon)-\bar{P}\bar{Q}\, \cos(v_k)) = \bar{Q}(2i\, \sinh(\epsilon)+\bar{P}\bar{Q})(\sin(v_k-i\epsilon)+PQ\, \cos(v_k)). \tag{3.120}$$

Using relations (3.99) and rearranging the terms, the previous equation becomes:

$$\begin{aligned} &\left(\frac{1}{\bar{Q}(\zeta_1)} - \frac{1}{\bar{Q}(\bar{\zeta_1})}\right) \cosh(\epsilon)\sin(v_k) + i\left(\frac{1}{\bar{Q}(\zeta_1)} + \frac{1}{\bar{Q}(\bar{\zeta_1})}\right) \cos(v_k)\sinh(\epsilon) = \\ &= (Q(\zeta_1) - Q(\bar{\zeta_1}))\cosh(\epsilon)\sin(v_k) + i\cos(v_k)\sinh(\epsilon)(Q(\zeta_1) - Q(\bar{\zeta_1})). \end{aligned} \tag{3.121}$$

Note that the relations (3.119) gives $\gamma^2(\zeta_1) = \overline{\gamma^2(\bar{\zeta_1})}$: this implies that the coefficients of the series of $\gamma^2(\lambda)$ with respect to $\lambda$ are real, consistently with the expansion (3.65). The choice:

$$\gamma(\zeta_1) = -\bar{\gamma}(\bar{\zeta_1}), \tag{3.122}$$

entails:

$$\bar{Q}(\zeta_1) = -\frac{1}{Q(\bar{\zeta_1})}. \tag{3.123}$$

With this constraint the equation (3.121) holds too. Moreover (3.122) makes the diagonal terms in $D_k D_k^\dagger$ equal. This shows that, under the given assumptions, $D_k$ is a unitary matrix.

### 3.2.4 Interpolating Hamiltonian flow

The aim of this subsection is to obtain the interpolating Hamiltonian flow of the discrete dynamics defined by the Bäcklund transformations (3.102a), (3.102b), (3.102c). I assume that the constraints on the parameters (3.117) still holds. Again the parameter $\epsilon$ plays the role of the time step and one obtains a one parameter ($\lambda_0$) family of interpolating flows and again it is possible to choose the value of the parameter so to discretize the continuous flow corresponding to each Hamiltonian $H_i$ in (3.66). To clarify the above points, let me take the limit $\epsilon \to 0$.
The functions $P$ and $Q$ given by (3.99) goes into:

$$Q = \frac{A_t(\lambda_0) - \gamma(\lambda_0)}{B_t(\lambda_0)} + O(\epsilon) \equiv Q_0 + O(\epsilon), \tag{3.124}$$

$$P = -i\epsilon \frac{B_t(\lambda_0)}{\gamma(\lambda_0)} + O(\epsilon^2) \equiv i\epsilon P_0 + O(\epsilon^2) \tag{3.125}$$

and the dressing matrix becomes:

$$\begin{aligned} D(\lambda) &= k\,\sin(\lambda - \lambda_0)\mathbb{1} + \\ +i\epsilon k &\begin{pmatrix} \cos(\lambda - \lambda_0)(P_0Q_0 - 1) & P_0 \\ Q_0(2 - P_0Q_0) & \cos(\lambda - \lambda_0)(1 - P_0Q_0) \end{pmatrix} + O(\epsilon^2). \end{aligned} \tag{3.126}$$

Reorganizing the terms with the help of $P_0$ and $Q_0$ given in the equations (3.124) and (3.125), one arrives at the expression:

$$\begin{aligned} D(\lambda) &= k\,\sin(\lambda - \lambda_0)\mathbb{1} + \\ -\frac{i\epsilon k}{\gamma(\lambda_0)} &\begin{pmatrix} A_t(\lambda_0)\cos(\lambda - \lambda_0) & B_t(\lambda_0) \\ C_t(\lambda_0) & -A_t(\lambda_0)\cos(\lambda - \lambda_0) \end{pmatrix} + O(\epsilon^2). \end{aligned} \tag{3.127}$$

It is now straightforward to show that in the limit $\epsilon \to 0$ the equation of the map $\tilde{L}D = DL$ turns into the Lax equation for a continuous flow:

$$\dot{L}(\lambda) = [L(\lambda), M(\lambda, \lambda_0)], \tag{3.128}$$

where the time derivative is defined as:

$$\dot{L} = \lim_{\epsilon \to 0} \frac{\tilde{L} - L}{2\epsilon} \tag{3.129}$$

and the matrix $M(\lambda, \lambda_0)$ has the form

$$\frac{i}{2\gamma(\lambda_0)} \begin{pmatrix} A_t(\lambda_0)\cot(\lambda - \lambda_0) & \frac{B_t(\lambda_0)}{\sin(\lambda - \lambda_0)} \\ \frac{C_t(\lambda_0)}{\sin(\lambda - \lambda_0)} & -A_t(\lambda_0)\cot(\lambda - \lambda_0) \end{pmatrix}. \tag{3.130}$$

The system (3.128) can be cast in Hamiltonian form:

$$\dot{L}(\lambda) = \{\mathcal{H}(\lambda_0), L(\lambda)\}, \tag{3.131}$$

with the Hamilton's function given by:

$$\mathcal{H}(\lambda_0) = \gamma(\lambda_0) = \sqrt{A_t^2(\lambda_0) + B_t(\lambda_0)C_t(\lambda_0)}. \tag{3.132}$$

Quite remarkably the Hamiltonian (3.132) characterizing the interpolating flow is (the square root of) the generating function (3.65) of the whole set of conserved quantities. By choosing the parameter $\lambda_0$ to be equal to any of the poles $(\lambda_i)$ of the Lax matrix, the map leads to $N$ different maps $\{BT^{(i)}\}_{i=1..N}$, where $BT^{(i)}$ discretizes the flow corresponding to the Hamiltonian $H_i$, given by equation (3.66).
More explicitly, proceeding as in the rational case and posing $\lambda_0 = \delta + \lambda_i$, in the limit $\delta \to 0$ the Hamilton's function (3.132) gives:

$$\gamma(\lambda_0) = \frac{s_i}{\delta} + \frac{H_i}{2s_i} + O(\delta) \tag{3.133}$$

and the equations of motion take the form:

$$\dot{L}(\lambda) = \frac{1}{2s_i}\{H_i, L(\lambda)\}. \tag{3.134}$$

Accordingly, the interpolating flow encompasses all the commuting flows of the system, so that the Bäcklund transformations turn out to be an *exact* time-discretizations of such interpolating flow.

### 3.2.5 Numerics

The figures report an example of the iteration of the map (3.102a), (3.102b), (3.102c). For simplicity I take $N = 2$. The computations shows the first 1500 iterations: the plotted variables are the physical ones $(s_1^x, s_1^y, s_1^z)$. Only one of the two spins is shown, namely that labeled by the subscript "1".

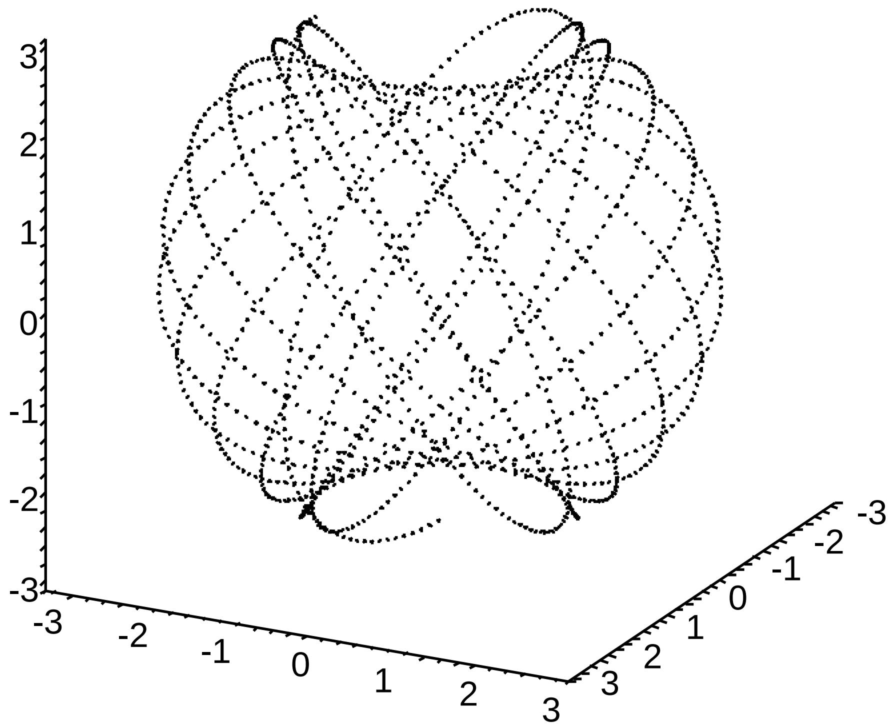


Figure 3.1: input parameters: $s_1^+ = 2 + i$, $s_1^- = 2 - i$, $s_1^3 = -2$, $s_2^+ = 50 + 40i$, $s_2^- = 50 - 40i$, $s_2^3 = 70$, $\lambda_1 = \pi/110$, $\lambda_2 = 7\pi/3$, $\lambda_0 = 0.1$, $\mu = -0.002i$.

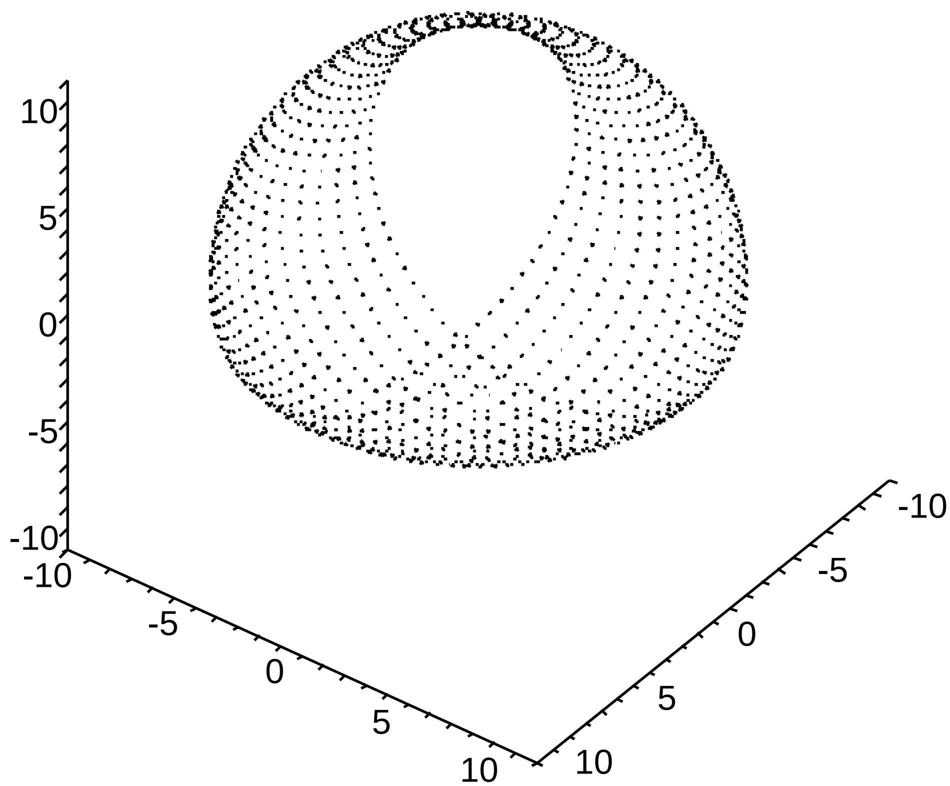


Figure 3.2: input parameters: $s_1^+ = 0.2 + 10i$, $s_1^- = 0.2 - 10i$, $s_1^3 = -1$, $s_2^+ = 10 - 30i$, $s_2^- = 10 + 30i$, $s_2^3 = 100$, $\lambda_1 = \pi$, $\lambda_2 = 7\pi/3$, $\lambda_0 = 0.1$, $\mu = -0.004i$.

## 3.3 The elliptic case

The elliptic Gaudin model is defined by the following Lax matrix:

$$L(\lambda) = \begin{pmatrix} A_e(\lambda) & B_e(\lambda) \\ C_e(\lambda) & -A_e(\lambda) \end{pmatrix}, \tag{3.135}$$

$$A_e(\lambda) = \sum_{j=1}^{N} \frac{\mathrm{cn}(\lambda-\lambda_j)}{\mathrm{sn}(\lambda-\lambda_j)} s_j^3, \qquad B_e(\lambda) = \sum_{j=1}^{N} \frac{s_j^1 - i s_j^2 \mathrm{dn}(\lambda-\lambda_j)}{\mathrm{sn}(\lambda-\lambda_j)} \tag{3.136}$$

$$C_e(\lambda) = \sum_{j=1}^{N} \frac{s_j^1 + i s_j^2 \mathrm{dn}(\lambda-\lambda_j)}{\mathrm{sn}(\lambda-\lambda_j)}.$$

As for the rational and trigonometric cases, since the beginning I suppose that the $N$ parameters of the model $\lambda_j$ are real. The Poisson structure is defined by the linear $r$-matrix structure [113]:

$$\big\{L(\lambda), L(\mu)\big\} = \big[r_e(\lambda-\mu), L(\lambda)\otimes \mathbb{1} + \mathbb{1}\otimes L(\mu)\big], \tag{3.137}$$

where $r_e$ is the elliptic $r$-matrix [35]:

$$r_e(\lambda) = \frac{i}{\mathrm{sn}(\lambda)} \begin{pmatrix} \mathrm{cn}(\lambda) & 0 & 0 & \frac{1-\mathrm{dn}(\lambda)}{2} \\ 0 & 0 & \frac{1+\mathrm{dn}(\lambda)}{2} & 0 \\ 0 & \frac{1+\mathrm{dn}(\lambda)}{2} & 0 & 0 \\ \frac{1-\mathrm{dn}(\lambda)}{2} & 0 & 0 & \mathrm{cn}(\lambda) \end{pmatrix}, \tag{3.138}$$

Hereafter, when there is no possibility of confusion, the modulus of the elliptic functions $k$ will be omitted for simplicity. The $r$-matrix structure (3.137) entails the following Poisson brackets for the functions (3.136):

$$\begin{aligned}
&\{A_e(\lambda), A_e(\mu)\} = 0, \\
&\{B_e(\lambda), B_e(\mu)\} = i(A_e(\lambda)+A_e(\mu))\frac{\mathrm{dn}(\lambda-\mu)-1}{\mathrm{sn}(\lambda-\mu)}, \\
&\{C_e(\lambda), C_e(\mu)\} = i(A_e(\lambda)+A_e(\mu))\frac{1-\mathrm{dn}(\lambda-\mu)}{\mathrm{sn}(\lambda-\mu)}, \\
&\{A_e(\lambda), B_e(\mu)\} = i\frac{C_e(\lambda)(1-\mathrm{dn}(\lambda-\mu)) - B_e(\lambda)(1+\mathrm{dn}(\lambda-\mu)) + 2B_e(\mu)\mathrm{cn}(\lambda-\mu)}{2\mathrm{sn}(\lambda-\mu)}, \\
&\{A_e(\lambda), C_e(\mu)\} = i\frac{B_e(\lambda)(\mathrm{dn}(\lambda-\mu)-1) + C_e(\lambda)(1+\mathrm{dn}(\lambda-\mu)) - 2C_e(\mu)\mathrm{cn}(\lambda-\mu)}{2\mathrm{sn}(\lambda-\mu)}, \\
&\{B_e(\lambda), C_e(\mu)\} = i\frac{(A_e(\mu)-A_e(\lambda))(1+\mathrm{dn}(\lambda-\mu))}{\mathrm{sn}(\lambda-\mu)}.
\end{aligned} \tag{3.139}$$

Equivalently, the dynamical variables $\left(s_j^1, s_j^2, s_j^3\right)$, $j = 1..N$, have to obey to the spin algebra:

$$\left\{s_j^a, s_k^b\right\} = \varepsilon_{abc}\delta_{jk}s_k^c, \tag{3.140}$$

where $\varepsilon_{abc}$ is the Levi-Civita symbol. Because of the direct sum structure of the Poisson bracket (3.140), the square length of each spin is a Casimir function for the elliptic Gaudin model, so one has a total of $N$ Casimirs, given by:

$$(s_j^1)^2 + (s_j^2)^2 + (s_j^3)^2 \doteq c_j^2, \qquad j = 1..N.$$

There exist $N$ integrals of motion whose generating function is the determinant of the Lax matrix (see appendix A):

$$-\det(L) = A_e^2(\lambda) + B_e(\lambda)C_e(\lambda) = \sum_{i=1}^{N}\left(\frac{c_i^2}{\mathrm{sn}^2(\lambda-\lambda_i)} + 2\varpi H_i\zeta(\varpi(\lambda-\lambda_i))\right) - H_0, \tag{3.141}$$

where $\zeta$ is the Weierstraß zeta function, $\varpi = (e_1 - e_2)^{-\frac{1}{2}}$, $e_i = \wp(\frac{w_i}{2})$ and $(w_1, w_2)$ are the periods of the Weierstraß $\wp$ function (see A for the notations). The $N$ Hamiltonians $H_i$ are explicitly given by:

$$H_i = \sum_{k\neq i}^{N} \frac{s_i^3 s_k^3 \mathrm{cn}(\lambda_i-\lambda_k) + s_i^2 s_k^2 \mathrm{dn}(\lambda_i-\lambda_k) + s_i^1 s_k^1}{\mathrm{sn}(\lambda_i-\lambda_k)}. \tag{3.142}$$

Note that only $N-1$ among these Hamiltonians are independent, because $\sum_i H_i = 0$. The remaining integral $H_0$ is given by the formula:

$$\begin{aligned} H_0 =& \sum_{i,k}^{N}(s_i^3 s_k^3 \mathrm{dn}(\lambda_i-\lambda_k) + k^2 s_i^2 s_k^2 \mathrm{cn}(\lambda_i-\lambda_k)) + \\ &+ \sum_{\substack{k,i\\k\neq i}}^{N} \frac{a(\lambda_i-\lambda_k)\left(s_i^3 s_k^3 \mathrm{cn}(\lambda_i-\lambda_k) + s_i^2 s_k^2 \mathrm{dn}(\lambda_i-\lambda_k) + s_i^1 s_k^1\right)}{\mathrm{sn}(\lambda_i-\lambda_k)}, \\ & a(\lambda) \doteq \varpi\big(\zeta(\varpi\lambda) - \zeta(2\varpi\lambda)\big) - \frac{1}{\mathrm{sn}(2\lambda)}. \end{aligned} \tag{3.143}$$

Due to the existence of an $r$-matrix, the Hamiltonians $H_i$ are in involution for the Poisson bracket (3.140) [9]:

$$\{H_i, H_j\} = 0, \quad i, j = 0, \ldots, N-1. \tag{3.144}$$

The corresponding Hamiltonian flows are then given by:

$$\frac{ds_j^a}{dt_i} = \{H_i, s_j^a\}, \qquad a = 1, 2, 3, \qquad j = 1 \ldots N. \tag{3.145}$$

### 3.3.1 The dressing matrix and the explicit transformations

The rational and trigonometric Gaudin models are limiting cases of the elliptic one. As we have seen in the previous two sections, the dressing matrices for the Lax matrices of these models are given, respectively, by the elementary Lax matrix of the $XXX$ and $XXZ$ Heisenberg spin chain on the lattice. It is then natural to make the ansatz of the elementary Lax matrix of the $XYZ$ Heisenberg spin chain for the elliptic Gaudin model. Note however that, since $\tilde{L}$ has to enjoy the same symmetry properties as the Lax matrix (3.135), also the dressing matrix $D(\lambda)$ has to possess these symmetries. Let me first show what they are. The quasi-periodicity of the Jacobi elliptic functions listed in appendix A (see (A.4)) entails the following formulae for $L(\lambda)$ (see also [114]):

$$L(\lambda+2K)=\sigma_3 L(\lambda)\sigma_3, \qquad L(\lambda+2iK')=\sigma_1 L(\lambda)\sigma_1, \tag{3.146}$$

where $K$ and $K'$ are respectively the complete elliptic integral of the first kind and the complementary integral (A.5 in appendix A). The above points suggest to make the following ansätz for $D(\lambda)$

$$D(\lambda)=\mathscr{S}_0\mathbb{1}+\frac{i}{\mathrm{sn}(\lambda)}\big(\mathscr{S}_1\sigma_1+\mathrm{dn}(\lambda)\mathscr{S}_2\sigma_2+\mathrm{cn}(\lambda)\mathscr{S}_3\sigma_3\big). \tag{3.147}$$

This is exactly the one-site Lax matrix for the $xyz$ Heisenberg spin chain on the lattice [35]. The symmetries (3.146) are also preserved:

$$D(\lambda+2K)=\sigma_3 D(\lambda)\sigma_3 \qquad D(\lambda+2iK')=\sigma_1 D(\lambda)\sigma_1. \tag{3.148}$$

So far, $\mathscr{S}_i$, $i=0...3$, are four undetermined variables, but we are free to fix one of them because of the homogeneity of the equation that defines the Bäcklund transformations

$$\tilde{L}(\lambda)D(\lambda-\lambda_0)=D(\lambda-\lambda_0)L(\lambda). \tag{3.149}$$

The Lax matrix (3.135) has simple poles at the points $\lambda=\lambda_k \pmod{2K,2iK'}$, $k=1...N$; since the relation (3.149) is an equivalence between meromorphic functions (the elements of the matrices), we have to equate the residues at the poles on both sides. Thanks to the symmetries (3.146) and (3.148) it is enough to look only at the poles in $\lambda=\lambda_k,\ k=1..N$, that is:

$$\tilde{L}_k D_k = D_k L_k, \tag{3.150}$$

where

$$L_k=\begin{pmatrix} s_k^3 & s_k^1-is_k^2\\ s_k^1+is_k^2 & -s_k^3\end{pmatrix}, \qquad D_k=D(\lambda=\lambda_k). \tag{3.151}$$

Now, since in general equation (3.150) gives an implicit relationship between the untilded variables and the tilded ones, the spectrality property has to be used. In analogy

with the rational and trigonometric cases, I force the determinant of the Darboux matrix $D(\lambda)$ to have two nondynamical zeroes for two arbitrary values of the spectral parameter $\lambda$, that is for $\lambda = \lambda_0 \pm \mu$. This leaves only *two* undetermined variables in (3.147). As I'm going to show, the spectrality will fix these two variables, that I will call $P$ and $Q$ for a comparison with the trigonometric case, as functions of the untilded dynamical variables only, so that the maps defined by (3.150) will be explicit.
Summarizing, by taking for simplicity $\mathscr{S}_0 = 1$, imposing the constraints

$$\det(D(\lambda - \lambda_0))\Big|_{\lambda=\lambda_0\pm\mu} = 0$$

and choosing a suitable parametrization of the constraints, it is possible to write:

$$D(\lambda) = \begin{pmatrix} 1 + i\mathscr{S}_3\frac{\mathrm{cn}(\lambda)}{\mathrm{sn}(\lambda)} & \frac{i\mathscr{S}_1+\mathscr{S}_2\,\mathrm{dn}(\lambda)}{\mathrm{sn}(\lambda)} \\ \frac{i\mathscr{S}_1-\mathscr{S}_2\,\mathrm{dn}(\lambda)}{\mathrm{sn}(\lambda)} & 1 - i\mathscr{S}_3\frac{\mathrm{cn}(\lambda)}{\mathrm{sn}(\lambda)} \end{pmatrix}, \quad \text{with} \begin{cases} i\mathscr{S}_3 = \dfrac{PQ - \mathrm{sn}(\mu)}{\mathrm{cn}(\mu)} \\ i\mathscr{S}_1 + \mathscr{S}_2\,\mathrm{dn}(\mu) = P \\ i\mathscr{S}_1 - \mathscr{S}_2\,\mathrm{dn}(\mu) = Q(2\,\mathrm{sn}(\mu) - PQ). \end{cases} \tag{3.152}$$

I recall again that $P$ and $Q$ are undetermined dynamical variables and that $\lambda_0$ and $\mu$ are constants: they are parameters for the Bäcklund transformations. Note also that with this parametrization, in the limit $k \to 0$, where $k$ is the modulus of the elliptic functions, one obtains, up to a trivial multiplicative factor, the dressing matrix for the trigonometric Gaudin model 3.96. Now it is possible to use the spectrality property to find $P$ and $Q$ in terms of one set of variables only, the untilded ones. The matrices $(D(\lambda-\lambda_0))\Big|_{\lambda=\lambda_0+\mu}$ and $(D(\lambda-\lambda_0))\Big|_{\lambda=\lambda_0-\mu}$ are of rank one. Their respective kernels are denoted by $|\Omega_+\rangle$ and $|\Omega_-\rangle$ that, as repeatedly seen, are also eigenvectors of $L(\lambda_0+\mu)$ and $L(\lambda_0-\mu)$:

$$\tilde{L}(\lambda_0 \pm \mu)D(\pm\mu)|\Omega_\pm\rangle = 0 = D(\pm\mu)\left[L(\lambda_0 \pm \mu)|\Omega_\pm\rangle\right] \Longrightarrow L(\lambda_0 \pm \mu)|\Omega_\pm\rangle = \gamma_\pm|\Omega_\pm\rangle. \tag{3.153}$$

By viewing the generating function of the integrals (3.141) as a function of $\lambda$, let me define:

$$\gamma^2(\lambda) \doteq -\det(L(\lambda)) = A_e(\lambda)^2 + B_e(\lambda)C_e(\lambda), \tag{3.154}$$

where $A_e(\lambda)$, $B_e(\lambda)$ and $C_e(\lambda)$ are given by (3.136). Thus, the two eigenvalues are given by $\gamma_\pm = \gamma(\lambda)\Big|_{\lambda=\lambda_0\pm\mu}$. The two kernels $|\Omega_\pm\rangle$ are written in terms of the variables $P$ and $Q$, so that the eigenvectors relations (3.153) for $L(\lambda_0 \pm \mu)$ suffice to express them in terms of the elements of the Lax matrix of the untilded variables. Explicitly, the two kernels are given by:

$$|\Omega_+\rangle = \begin{pmatrix} 1 \\ -Q \end{pmatrix}, \qquad |\Omega_-\rangle = \begin{pmatrix} P \\ 2\,\mathrm{sn}(\mu) - PQ \end{pmatrix} \tag{3.155}$$

and these expressions in turn lead to the formulae:

$$Q = Q(\lambda_0 + \mu) = \left.\frac{A_e(\lambda) - \gamma(\lambda)}{B_e(\lambda)}\right|_{\lambda=\lambda_0+\mu}, \qquad \frac{1}{P} = \frac{Q(\lambda_0 + \mu) - Q(\lambda_0 - \mu)}{2\,\mathrm{sn}(\mu)}. \quad (3.156)$$

Note that, for an arbitrary number $N$ of interacting spins of the model, $P$ and $Q$ contain *all* the dynamical variables so that the Bäcklund transformations I'm going to write touch all the spin sites. These maps associate to a given solution of the equations of motion (3.145) a new solution. Given the initial conditions, the generating function (3.141), and therefore the function $\gamma(\lambda)$, is a constant independent of time. On the other hand, as it will be clear later, if $\gamma(\lambda)$ is constant, then the Bäcklund transformations are actually rational maps (together with their inverse), and as such, integrability detectors based on the algebraic entropy can be applied. This point, as all other features of the Bäcklund transformations for elliptic Gaudin model, are in common with the trigonometric and rational cases. It is clear indeed that in the limit $k \to 0$ all the results given in this section reduce to the trigonometric case of the last section that, in turn, as I showed explicitly, reduces to the rational one in the small angle limit. Given the rational character of the maps (having in mind to fix the initial conditions), I'm quite sure that the algebraic entropy, as defined for example in [17], is zero. Until now I have not verified this conjecture, but there are many facts that lead O. Ragnisco and me to this belief. Furthermore I'm quite sure that the maps are not only integrable but also explicitly solvable. What I mean with explicitly will be clear in the next chapter where I deal with an actual "integration" of a Bäcklund transformation. I postpone there further comments. The equation (3.150) allows to write the explicit transformations as follows:

$$\tilde{s}_k^1 = \frac{((\alpha_k^2 + \varsigma_k^2 - \beta_k^2 - \delta_k^2)s_k^1 + i(\delta_k^2 + \varsigma_k^2 - \beta_k^2 - \alpha_k^2)s_k^2 - 2(\alpha_k\beta_k - \varsigma_k\delta_k)s_k^3)}{2\Lambda_k}, \quad (3.157a)$$

$$\tilde{s}_k^2 = \frac{(i(\alpha_k^2 + \delta_k^2 - \varsigma_k^2 - \beta_k^2)s_k^1 + (\beta_k^2 + \varsigma_k^2 + \alpha_k^2 + \delta_k^2)s_k^2 - 2i(\beta_k\alpha_k + \varsigma_k\delta_k)s_k^3)}{2\Lambda_k}, \quad (3.157b)$$

$$\tilde{s}_k^3 = \frac{((\beta_k\varsigma_k - \alpha_k\delta_k)s_k^1 + i(\beta_k\varsigma_k + \alpha_k\delta_k)s_k^2 + (\alpha_k\varsigma_k + \beta_k\delta_k)s_k^3)}{\Lambda_k}, \quad (3.157c)$$

where, to simplify notations, I have introduced the functions $(\alpha_k, \beta_k, \delta_k, \Delta_k, \varsigma_k)$ defined by the following formulae:

$$\begin{aligned}
\alpha_k &= \operatorname{sn}(\lambda_k - \lambda_0) + \frac{PQ - \operatorname{sn}(\mu)}{\operatorname{cn}(\mu)}\operatorname{cn}(\lambda_k - \lambda_0),\\
\beta_k &= \frac{(P + Q(2\operatorname{sn}(\mu) - PQ))}{2} + \frac{(P - Q(2\operatorname{sn}(\mu) - PQ))}{2\operatorname{dn}(\mu)}\operatorname{dn}(\lambda_k - \lambda_0),\\
\delta_k &= \frac{(P + Q(2\operatorname{sn}(\mu) - PQ))}{2} - \frac{(P - Q(2\operatorname{sn}(\mu) - PQ))}{2\operatorname{dn}(\mu)}\operatorname{dn}(\lambda_k - \lambda_0),\\
\varsigma_k &= \operatorname{sn}(\lambda_k - \lambda_0) - \frac{PQ - \operatorname{sn}(\mu)}{\operatorname{cn}(\mu)}\operatorname{cn}(\lambda_k - \lambda_0),\\
\Lambda_k &= \alpha_k\varsigma_k - \beta_k\delta_k.
\end{aligned} \tag{3.158}$$

When the elliptic modulus $k$ of the Jacobi elliptic functions is zero, the transformations (3.157a), (3.157b), (3.157c) coincide with those for the trigonometric Gaudin magnet (3.102a), (3.102b), (3.102c): to make a direct comparison the only care is to express the trigonometric maps in terms of the variables $(s_j^1, s_j^2, s_j^3)$, or vice versa the elliptic transformations in terms of $(s_j^+, s_j^-, s_j^3)$.

## 3.3.2 Canonicity

At this point one has to deal with the canonicity of the maps (3.157a), (3.157b), (3.157c). Again it is possible to follow the Sklyanin's arguments of the previous two cases, the rational and trigonometric. I briefly recall the method, that is however reported in more details in subsection 3.1.3. Consider the relation (3.149) in an extended phase space, whose coordinate are given by $(s_k^1, s_k^2, s_k^3, P, Q)$ and suppose that $D(\lambda)$ obeys the *quadratic* Poisson bracket, as follows:

$$\{D(\lambda) \otimes \mathbb{1}, \mathbb{1} \otimes D(\tau)\} = [r_e(\lambda - \tau), D(\lambda) \otimes D(\tau)]. \tag{3.159}$$

In the extended space one has to re-define (3.149) as:

$$\tilde{L}(\lambda)\tilde{D}(\lambda - \lambda_0) = D(\lambda - \lambda_0)L(\lambda). \tag{3.160}$$

Obviously, in the left hand side of the previous equation one has to use tilded variables also for $D(\lambda)$ because (3.160) defines the Bäcklund transformation in the extended phase space, where there is also a $\tilde{P}$ and a $\tilde{Q}$. The key observation is that in the extended phase space the entries of $D$ Poisson commutes with those of $L$. In fact, since both $L$ and $D$ have the same Poisson structure, given by equation (3.159), *and* the entries of $L$ and $D$ Poisson commute, then this property holds true for $LD$ and $DL$ as well. This means that the transformation (3.160) defines a canonical transformation in the extended phase space. Sklyanin was able to show [111] (see also subsection 3.1.3) that, if one now restricts the variables to the constrained manifold $\tilde{P} = P$ and $\tilde{Q} = Q$, the symplecticity is preserved; however this constraint leads to a dependence of $P$ and $Q$ on the entries of $L$, that for consistency must be the same as the one given by the equation (3.160) on this constrained manifold. But there (3.160) reduces to (3.149), so that the map preserves the spectrum of $L(\lambda)$ and is canonical. What remains to show is that indeed (3.159) is fulfilled by our $D(\lambda)$. Recall that for the Bäcklund transformations of the rational Gaudin magnet the dressing matrix is equipped with the quadratic Poisson structure imposed by the rational $r$-matrix provided $P$ and $Q$ are canonically conjugated in the extended space. In the trigonometric case one needs a non trivial bracket between $P$ and $Q$ in the extended space to guarantee the simplecticity of Bäcklund transformations. In the elliptic case one finds a non trivial bracket that, in the limit $k \to 0$ goes to the trigonometric result as given in( 3.113). By a direct inspection it is possible to show that (3.106) entails the following brackets between the elements $\mathscr{S}_i$, $i = 0...3$ [35] (see also (2.14)):

$$\begin{aligned} \{\mathscr{S}_i, \mathscr{S}_0\} &= iJ_{jk}\mathscr{S}_j\mathscr{S}_k, \\ \{\mathscr{S}_i, \mathscr{S}_j\} &= -i\mathscr{S}_0\mathscr{S}_k, \end{aligned} \tag{3.161}$$

where $(i, j, k)$ is a cyclic permutation of $(1, 2, 3)$ with $J_{12} = k^2$, $J_{23} = 1 - k^2$, $J_{31} = -1$. With the following positions:

$$\begin{cases} (\mathscr{S}_0)^2 = \dfrac{\mathrm{cn}(\mu)\mathrm{dn}(\mu)}{\mathrm{sn}(\mu)\left(1 - 2\,\mathrm{sn}(\mu)PQ + P^2Q^2 - k^2\left[(PQ - \mathrm{sn}(\mu))^2 + \mathrm{cn}(\mu)^2\frac{(Q(PQ-2\,\mathrm{sn}(\mu))-P)^2}{4}\right]\right)}, \\ i\mathscr{S}_3 = \dfrac{PQ - \mathrm{sn}(\mu)}{\mathrm{cn}(\mu)}\mathscr{S}_0, \\ i\mathscr{S}_1 + \mathscr{S}_2\,\mathrm{dn}(\mu) = P\mathscr{S}_0, \\ i\mathscr{S}_1 - \mathscr{S}_2\,\mathrm{dn}(\mu) = Q(2\,\mathrm{sn}(\mu) - PQ)\,\mathscr{S}_0, \end{cases} \tag{3.162}$$

after some calculations one can show that indeed (3.161) are fullfilled provided that:

$$\{Q, P\} = i\frac{\left(1 - 2\,\mathrm{sn}(\mu)PQ + P^2Q^2 - k^2\left[(PQ - \mathrm{sn}(\mu))^2 + \mathrm{cn}(\mu)^2\frac{(Q(PQ-2\,\mathrm{sn}(\mu))-P)^2}{4}\right]\right)}{\mathrm{cn}(\mu)\mathrm{dn}(\mu)}. \tag{3.163}$$

So the simplecticity of the maps is proved. At this point let us to make some remarks. Firstly, up to the multiplicative factor $\mathscr{S}_0$, the dressing matrix defined by the relations (3.162) is completely equivalent to the dressing matrix as given by the equation (3.152). As explained before, due to the homogeneity of the equation (3.149), a proportionality factor between two dressing matrices is inessential as far as Bäcklund transformations are concerned, so by this point of view the definitions (3.162) are compatible with

(3.152). Secondly, the Poisson bracket (3.163) between $P$ and $Q$ reduces , in the limit $k \to 0$, to the bracket of the corresponding variables in the trigonometric case.

### 3.3.3 Physical Bäcklund transformations

Also for the more general elliptic Bäcklund transformations it is possible to map real variables to real variables; the constraints that must be taken in the parameter of the transformations are the same. In fact the proper choice amounts to require $\lambda_0$ to be a real number and $\mu$ to be a purely imaginary number. So, hereafter in this section, I put:

$$\mu = i\epsilon, \qquad (\lambda_0, \epsilon) \in \mathbb{R}^2. \tag{3.164}$$

The matrices $L_k$, $k = 1..N$ defined in (3.151) and corresponding to the real solutions $\left(s_j^1, s_j^2, s_j^3\right)$ of the equations of motion are Hermitian. The request for physical Bäcklund transformations is equivalent to the request that the dressed matrices $\tilde{L}_k$ be Hermitian as well. By (3.150) one sees that this means to have dressing matrices $D_k$ proportional to unitary matrices. I claim that indeed, when (3.164) are fulfilled, then $D_k$ are of the form:

$$D_k = \begin{pmatrix} \rho_k & \varrho_k \\ -\bar{\varrho}_k & \bar{\rho}_k \end{pmatrix}, \tag{3.165}$$

where the bar means complex conjugation. For clarity let me make the following positions:

$$\lambda_+ \doteq \lambda_0 + i\epsilon, \qquad \lambda_- \doteq \bar{\lambda}_+. \tag{3.166}$$

The functions $A_e(\lambda), B_e(\lambda), C_e(\lambda)$, as defined in (3.136), verify:

$$A_e(\lambda_+) = \bar{A}_e(\lambda_-), \qquad B_e(\lambda_+) = \bar{C}_e(\lambda_-), \qquad C_e(\lambda_+) = \bar{B}_e(\lambda_-). \tag{3.167}$$

These relations entail $\gamma^2(\lambda_+) = \bar{\gamma}^2(\lambda_-)$, meaning that the coefficients of the series of $\gamma^2(\lambda)$ with respect to $\lambda$ are real, consistently with the expansion (3.141). Recall also that the matrices $D_k$ are written in terms of $P$ and $Q$, that are defined by the relations

$$Q = Q(\lambda_+) = \frac{A_e(\lambda_+) - \gamma(\lambda_+)}{B_e(\lambda_+)} = -\frac{C_e(\lambda_+)}{A_e(\lambda_+) + B_e(\lambda_+)}, \qquad P = \frac{2\,\mathrm{sn}(i\epsilon)}{Q(\lambda_+) - Q(\lambda_-)}. \tag{3.168}$$

Now the choice $\gamma(\lambda_+) = -\bar{\gamma}(\lambda_-)$ entails

$$\bar{Q}(\lambda_+) = -\frac{1}{Q(\lambda_-)}$$

and this equation in turn implies that the matrices $D_k = D(\lambda)|_{\lambda=\lambda_k}$, with $D(\lambda)$ given by (3.152), are of the form (3.165), with $\rho_k$ and $\varrho_k$ given by the following formulae:

$$\begin{aligned} \rho_k &= 1 + \frac{\mathrm{sn}(i\epsilon)\,\mathrm{cn}(\lambda_k)}{\mathrm{cn}(i\epsilon)\,\mathrm{sn}(\lambda_k)} \frac{|Q|^2-1}{|Q|^2+1}, \\ \varrho_k &= \frac{\mathrm{sn}(i\epsilon)}{\mathrm{sn}(\lambda_k)} \left( \frac{\bar{Q}+Q}{|Q|^2+1} + \frac{\mathrm{dn}(\lambda_k)}{\mathrm{dn}(i\epsilon)} \frac{\bar{Q}-Q}{|Q|^2+1} \right). \end{aligned} \tag{3.169}$$

So, under the given assumptions, the matrices $D_k$ are indeed proportional to unitary matrices.

### 3.3.4 Interpolating Hamiltonian flow

Now I want to get the interpolating flow of the discrete dynamics generated by the maps (3.157a), (3.157b), (3.157b). Given all the symmetries between the rational, trigonometric and elliptic maps, there can only be little doubts about the fact that the Bäcklund transformation can be seen as a time discretization of a one-parameter ($\lambda_0$) family of Hamiltonian flows with the difference $2\epsilon$ playing the role of the time-step and with the Hamiltonian defining the interpolating flow given by $\gamma(\lambda_0)$, where $\gamma(\lambda)$ is defined in (3.154). To remove any suspect, I will show explicitly the truth of the previous assertions. First of all let me take the limit $\epsilon \to 0$.
One has:

$$Q = \frac{A_e(\lambda_0) - \gamma(\lambda_0)}{B_e(\lambda_0)} + O(\epsilon), \tag{3.170}$$

$$P = -i\epsilon \frac{B_e(\lambda_0)}{\gamma(\lambda_0)} + O(\epsilon^2). \tag{3.171}$$

One can carefully insert these expressions in the dressing matrix (3.152) to find:

$$D(\lambda - \lambda_0) = \mathbb{1} - \frac{i\epsilon}{\gamma(\lambda_0)\mathrm{sn}(\lambda-\lambda_0)} D_0(\lambda, \lambda_0), \tag{3.172}$$

where

$$D_0(\lambda,\lambda_0) \doteq \begin{pmatrix} A_e(\lambda_0)\mathrm{cn}(\lambda-\lambda_0) & \frac{B_e(\lambda_0)+C_e(\lambda_0)}{2} + \frac{B_e(\lambda_0)-C_e(\lambda_0)}{2}\mathrm{dn}(\lambda-\lambda_0) \\ \frac{B_e(\lambda_0)+C_e(\lambda_0)}{2} - \frac{B_e(\lambda_0)-C_e(\lambda_0)}{2}\mathrm{dn}(\lambda-\lambda_0) & -A_e(\lambda_0)\mathrm{cn}(\lambda-\lambda_0) \end{pmatrix}. \tag{3.173}$$

In the limit $\epsilon \to 0$ the equation of the map $\tilde{L}D = DL$ turns into the Lax equation for a continuous flow:

$$\dot{L}(\lambda) = [L(\lambda), M(\lambda,\lambda_0)]. \tag{3.174}$$

where the time derivative is defined as:

$$\dot{L} = \lim_{\epsilon\to 0} \frac{\tilde{L} - L}{2\epsilon} \tag{3.175}$$

and the matrix $M(\lambda, \lambda_0)$ is given by:

$$M(\lambda, \lambda_0) = \frac{i}{2\gamma(\lambda_0)\mathrm{sn}(\lambda - \lambda_0)} D_0(\lambda, \lambda_0). \tag{3.176}$$

With the help of the Poisson brackets between the elements of the Lax matrix (3.139), the dynamical system (3.174) can be cast in Hamiltonian form:

$$\dot{L}(\lambda) = \{\mathcal{H}(\lambda_0), L(\lambda)\}, \tag{3.177}$$

with the Hamilton's function given by:

$$\mathcal{H}(\lambda_0) = \gamma(\lambda_0) = \sqrt{A_e^2(\lambda_0) + B_e(\lambda_0)C_e(\lambda_0)}. \tag{3.178}$$

So the Hamiltonian (3.178) characterizing the interpolating flow is (the square root of) the generating function (3.141) of the whole set of conserved quantities. By choosing the parameter $\lambda_0$ to be equal to any of the poles $(\lambda_i)$ of the Lax matrix, the map leads to $N$ different maps $\{BT^{(i)}\}_{i=1..N}$, where $BT^{(i)}$ discretizes the flow corresponding to the Hamiltonian $H_i$, given by equation (3.142). In fact, by posing $\lambda_0 = \delta + \lambda_i$ and taking the limit $\delta \to 0$, the Hamilton's function (3.178) gives:

$$\gamma(\lambda_0) = \frac{c_i}{\delta} + \frac{H_i}{c_i} + O(\delta). \tag{3.179}$$

and the equations of motion take the form:

$$\dot{L}(\lambda) = \frac{1}{c_i}\{H_i, L(\lambda)\} \tag{3.180}$$

In the previous two chapters I proved the same results for the trigonometric and rational cases, but they can be easily inherited from the more general elliptic structures with the usual limiting procedures.

# Chapter 4

# An application to the Kirchhoff top

In this chapter, as an application of the results found for the trigonometric Gaudin magnet in [93], [94] and of the technique of pole coalescence [80], [86], see also section 2.4, I will construct the Bäcklund transformations for a particular case of the Clebsch model, the Kirchhoff top.

The Clebsch model (C.19) is an integrable case of the Kirchhoff equations [59] (see equations (C.17) in the appendix [C]) describing the motion of a solid in infinite incompressible fluid. The physical problem with the derivation of the equations of motion are described and given in appendix C; for the details the reader is referred there.

The total kinetic energy of the system *solid* + *fluid* is given by the expression (C.22):

$$T = \frac{1}{2}\Big(\frac{p_1^2+p_2^2}{A_1}+\frac{p_3^2}{A_3}\Big)+\frac{1}{2}\Big(\frac{J_1^2+J_2^2}{B_1}+\frac{J_3^2}{B_3}\Big). \tag{4.1}$$

The vectors $(p_1, p_2, p_3)$ and $(J_1, J_2, J_3)$ are respectively the components of the total impulse and total angular momentum of the system, i.e. the sum of the impulse and angular momentum of the solid and those applied by the solid to the boundary of the fluid in contact with it. The four constants $A_1, A_3, B_1, B_3$ depend on the shape of the solid. The impulse and angular momentum must obey to the Lie-Poisson $e(3)$ algebra given by the following Poisson brackets:

$$\{J_i, J_j\} = \varepsilon_{ijk}J_k, \quad \{J_i, p_j\} = \varepsilon_{ijk}p_k, \quad \{p_i, p_j\} = 0. \tag{4.2}$$

The evolution equations for the impulse $\mathbf{p} = (p_1, p_2, p_3)$ and angular momentum $\mathbf{J} = (J_1, J_2, J_3)$ are then given by:

$$\dot{\mathbf{p}} = \{T, \mathbf{p}\}, \qquad \dot{\mathbf{J}} = \{T, \mathbf{J}\}. \tag{4.3}$$

## 4.1 Contraction of two sites trigonometric Gaudin model

In this Section I show how to obtain the Lax matrix of the Kirchhoff top by a procedure of poles coalescence on the Lax matrix of two-sites trigonometric Gaudin model. I will follow [95]. The two-sites Lax matrix is obviously given by:

$$L_t(\lambda) = \begin{pmatrix} A_t(\lambda) & B_t(\lambda) \\ C_t(\lambda) & -A_t(\lambda) \end{pmatrix}, \tag{4.4}$$

$$A_t(\lambda) = \sum_{j=1}^{2} \cot(\lambda - \lambda_j) s_j^3, \quad B_t(\lambda) = \sum_{j=1}^{2} \frac{s_j^-}{\sin(\lambda - \lambda_j)}, \quad C_t(\lambda) = \sum_{j=1}^{2} \frac{s_j^+}{\sin(\lambda - \lambda_j)}. \tag{4.5}$$

In (4.4) and (4.5) $\lambda_1$ and $\lambda_2$ are the two arbitrary real parameters of the two-sites Gaudin model that in the follow I will let to coalesce in order to obtain the Kirchhoff top. The Poisson brackets and the $r$-matrix structure are described by (3.60) and (3.63). The integrals of motion are inferred from their generating function, that is the determinant of the Lax matrix (4.4):

$$-\det(L_t(\lambda)) = \frac{C_{1t}}{\sin(\lambda - \lambda_1)^2} + \frac{C_{2t}}{\sin(\lambda - \lambda_2)^2} + \frac{H_t \sin(\lambda_1 - \lambda_2)}{\sin(\lambda - \lambda_1)\sin(\lambda - \lambda_2)} - H_0^2, \tag{4.6}$$

where $C_{1t}$ and $C_{2t}$ are the Casimirs of the algebra (3.60) given by $C_{it} = (s_i^3)^2 + s_i^+ s_i^-$, $i = 1, 2$, while the integrals of motion $H_t$ and $H_0$ are:

$$\begin{aligned} H_t &= \frac{2\cos(\lambda_1 - \lambda_2) s_1^3 s_2^3 + s_1^+ s_2^- + s_1^- s_2^+}{\sin(\lambda_1 - \lambda_2)}, \\ H_0 &= s_1^3 + s_2^3 \doteq J_t^3. \end{aligned} \tag{4.7}$$

The two integrals (4.7) are in involution with respect to the Poisson bracket (3.60), i.e. $\{H_t, H_0\} = 0$. As explained in section 2.4, to perform the poles coalescence one has to introduce the contraction parameter $\epsilon$ and to take in the Lax matrix (4.4) $\lambda_1 \to \epsilon\lambda_1$ and $\lambda_2 \to \epsilon\lambda_2$. For small $\epsilon$ the Taylor series on the Lax matrix reads:

$$L_G \to L_1 + \epsilon L_2 + O(\epsilon^2), \tag{4.8}$$

$$L_1 \doteq \begin{pmatrix} \cot(\lambda)(s_1^3 + s_2^3) & \frac{s_1^- + s_2^-}{\sin(\lambda)} \\ \frac{s_1^+ + s_2^+}{\sin(\lambda)} & -\cot(\lambda)(s_1^3 + s_2^3) \end{pmatrix},$$

$$L_2 \doteq \begin{pmatrix} \frac{\lambda_1 s_1^3 + \lambda_2 s_2^3}{\sin(\lambda)^2} & \frac{\cot(\lambda)(\lambda_1 s_1^- + \lambda_2 s_2^-)}{\sin(\lambda)} \\ \frac{\cot(\lambda)(\lambda_1 s_1^+ + \lambda_2 s_2^+)}{\sin(\lambda)} & -\frac{\lambda_1 s_1^3 + \lambda_2 s_2^3}{\sin(\lambda)^2} \end{pmatrix}.$$

The Lax matrix for the Kirchhoff top is unveiled by identifying $\mathbf{J}$ and $\mathbf{p}$ with the following vectors (I recall that the notation is $v_i^{\pm} = v_i^1 \pm i v_i^2$, $\mathbf{v}_i = (v_i^1, v_i^2, v_i^3)$ for any vector set $\mathbf{v}_i$):

$$\mathbf{J} \doteq \mathbf{s}_1 + \mathbf{s}_2, \qquad \mathbf{p} \doteq \epsilon(\lambda_1 \mathbf{s}_1 + \lambda_2 \mathbf{s}_2). \tag{4.9}$$

By a direct calculation it is easy to see, by using (3.60), that the Poisson brackets for the variables $\mathbf{J}$ and $\mathbf{p}$, as defined by (4.9), coincide with the expressions (4.2). The Lax matrix then reads as:

$$L(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix} = \begin{pmatrix} \cot(\lambda) J^3 + \frac{p^3}{\sin(\lambda)^2} & \frac{J^-}{\sin(\lambda)} + \frac{\cot(\lambda) p^-}{\sin(\lambda)} \\ \frac{J^+}{\sin(\lambda)} + \frac{\cot(\lambda) p^+}{\sin(\lambda)} & -(\cot(\lambda) J^3 + \frac{p^3}{\sin(\lambda)^2}) \end{pmatrix}. \tag{4.10}$$

The determinant of this matrix is the generating function of the integrals of motions; in fact the following formula holds:

$$-\det(L(\lambda)) \doteq \gamma^2(\lambda) = \frac{2H_1}{\sin(\lambda)^2} + 2H_0 \cot(\lambda)^2 + 2C_2 \frac{\cot(\lambda)^2}{\sin(\lambda)^2} + 2C_1 \frac{\cot(\lambda)}{\sin(\lambda)^2}, \tag{4.11}$$

where $C_1$ and $C_2$ are the two Casimirs:

$$\sum_{i=1}^{3} p_i J_i \doteq C_1, \qquad \sum_{i=1}^{3} p_i^2 \doteq 2C_2. \tag{4.12}$$

while $H_0$ and $H_1$ are the two commuting integrals given by:

$$H_1 = \frac{1}{2}(J_1^2 + J_2^2 + p_3^2), \qquad 2H_0 = J_3^2, \qquad \{H_1, H_0\} = 0. \tag{4.13}$$

In all the cases for which the equivalence $B_1^{-1} = A_3^{-1} - A_1^{-1}$ between the coefficients in (4.1) holds, it is possible to rewrite the total kinetic energy (4.1) in terms of the quantities (4.12), (4.13):

$$T = \frac{C_2}{A_1} + \frac{H_0}{B_3} + \frac{H_1}{B_1}. \tag{4.14}$$

## 4.2 Separation of variables

It is possible to perform the separation of variables for the previous model using the so called Sklyanin's magic recipe, i.e. taking as dynamical variables the poles of the normalized Baker-Akhiezer function and the corresponding eigenvalues of the Lax matrix (see subsection 1.5.2). As normalization vector I take the "standard" one, i.e. that given by $\alpha = (1, 0)$. The equation for the Baker-Akhiezer function reads:

$$L(\lambda)\psi = \gamma\psi, \qquad \alpha \cdot \psi = 1, \tag{4.15}$$

where $(\lambda, \gamma)$ belong to the spectral curve defined by $\det(L(\lambda) - \gamma \mathbb{1}) = 0$. The equation for the pole of $\psi$ is:

$$(1,0)(L(\lambda) - \gamma \mathbb{1})^{\dagger}\big|_{\lambda = x_1} = 0, \tag{4.16}$$

where the dag denotes the matrix of cofactors. The separation variables are then:

$$B(x_1) = 0 \to x_1 = -\arctan(\frac{p^-}{J^-}), \qquad \gamma\big|_{\lambda=x_1} \doteq iy_1 = -\frac{J^-}{p^-}(J^3 - \frac{J^-}{p^-}p^3) + p^3. \tag{4.17}$$

The variables $(y_1, x_1)$ are canonical with respect to the Poisson brackets (4.2). In order to separate the variables one needs of another pair of canonical variables. I take:

$$x_2 = -\text{arctanh}\Big(\frac{(J^-)^2 + (p^-)^2 - 1}{(J^-)^2 + (p^-)^2 + 1}\Big), \qquad y_2 = iJ^3. \tag{4.18}$$

The new variables $(y_i, x_i)$ are canonical:

$$\{x_i, x_j\} = \{y_i, y_j\} = 0, \qquad \{y_i, x_j\} = \delta_{ij}, \qquad i, j = 1, 2. \tag{4.19}$$

The map between the variables $(J^{\pm}, J^3)$, $(p^{\pm}, p^3)$ and the canonical ones read as follow:

$$\begin{aligned}
J^3 &= -iy_2, \\
J^- &= \cos(x_1)e^{-x_2}, \\
J^+ &= e^{x_2}\big(y_2\cos(x_1) + y_1\sin(x_1)\big)\big(y_2(1+\sin(x_1)^2) - y_1\cos(x_1)\sin(x_1)\big) + \\
&\quad - \frac{2e^{x_2}}{\sin(x_1)^2}\big(C_1\sin(x_1) + C_2\cos(x_1)\big), \\
p^3 &= i\sin(x_1)\big(y_2\cos(x_1) + y_1\sin(x_1)\big), \\
p^- &= -\sin(x_1)e^{-x_2}, \\
p^+ &= -\frac{e^{x_2}}{\sin(x_1)}\big(2C_2 + \sin(x_1)^2(y_2\cos(x_1) + y_1\sin(x_1))^2\big),
\end{aligned} \tag{4.20}$$

The separation equations are obtained rewriting the integrals (4.13) in terms of the canonical variables:

$$\begin{aligned}
H_1 &= \frac{y_2^2\cos(x_1)^2 - y_1^2\sin(x_1)^2}{2} - \cot(x_1)\big(C_1 + C_2\cot(x_1)\big), \\
H_0 &= -y_2^2.
\end{aligned} \tag{4.21}$$

The previous separation equations can be used to separate the Hamilton-Jacobi equation and to find the action in terms of elliptic integrals. It is possible however to obtain more compact expressions, furthermore in terms of only real variables, by integrating the model through Kirchhoff variables. This is the subject of the next section.

## 4.3 Integration of the model

Gustav Kirchhoff in his "*Vorlesungen über mathematische Physik*" [59] deals with the problem of integrating equations of motion (C.17). For the physical variables $(J^1, J^2, J^3)$ and $(p^1, p^2, p^3)$ in our case they are written in extended form as:

$$\begin{cases} \dot{p}^1(t) = \alpha J^3(t)p^2(t) - \beta J^2(t)p^3(t), \\ \dot{p}^2(t) = \beta J^1(t)p^3(t) - \alpha p^1(t)J^3(t), \\ \dot{p}^3(t) = \beta\big(J^2(t)p^1(t) - J^1(t)p^2(t)\big), \end{cases} \qquad \begin{cases} \dot{J}^1(t) = (\alpha-\beta)J^2(t)J^3(t) + \beta p^2(t)p^3(t), \\ \dot{J}^2(t) = (\beta-\alpha)J^1(t)J^3(t) - \beta p^1(t)p^3(t), \\ \dot{J}^3(t) = 0 \to J^3 = const \doteq M. \end{cases} \tag{4.22}$$

where for simplicity I have posed $\alpha \doteq B_3^{-1}$ and $\beta \doteq B_1^{-1}$. The new variables suggested by Kirchhoff are given by $p^1 = s\cos(f)$, $p^2 = s\sin(f)$, $J^1 = \sigma\cos(\psi + f)$, $J^2 = \sigma\sin(\psi + f)$. In terms of these variables the equations of motion can be written as:

$$\begin{cases} \dot{s}(\tau) = -\sigma(\tau)p^3(\tau)\sin(\psi(\tau)), \\ \dot{\sigma}(\tau) = -s(\tau)p^3(\tau)\sin(\psi(\tau)), \\ \dot{p}^3(\tau) = s(\tau)\sigma(\tau)\sin(\psi(\tau)), \\ \dot{\psi}(\tau) = M - p^3(\tau)\cos(\psi(\tau))\Big(\dfrac{\sigma(\tau)}{s(\tau)} + \dfrac{s(\tau)}{\sigma(\tau)}\Big), \\ \dot{f}(\tau) = \dfrac{\sigma(\tau)}{s(\tau)}p^3(\tau)\cos(\psi(\tau)) - \dfrac{\alpha}{\beta}M, \end{cases} \quad \text{where} \quad \begin{cases} \tau \doteq \beta t, \\ (\dot{\ }) \doteq \dfrac{\partial(\ )}{\partial\tau}. \end{cases} \tag{4.23}$$

By using the constraints given by the Casimirs and the integral $H_1$, i.e. $2H_1 = \sigma^2 + (p^3)^2$, $2C_2 = s^2 + (p^3)^2$, $s\sigma\cos(\psi) + Mp^3 = C_1$, one can readily obtains the equation for the evolution of $p^3$:

$$(\dot{p}^3)^2 = \Big((p^3)^2 - 2C_2\Big)\Big((p^3)^2 - 2H_1\Big) - \Big(Mp^3 - C_1\Big)^2. \tag{4.24}$$

At this point Kirchhoff notes that this equation is integrable and that one can obtain by the expression of $p^3$ those for the other variables, but soon after he passes to consider the special case $J^1 = J^3 = p^2 = 0$. At my knowledge the first author to integrate this system was G.E. Halphen in 1886 [47], also if some authors give Kirchhoff as reference for the complete integration of the equations of motion[1]. The expression for $p^3(t)$ of Halphen is written in terms of Weierstrass $\wp$ function; it reads as:

$$p^3(\tau) = \frac{1}{2}\Big(\frac{\dot{\wp}\big(\tau + K, \Phi + 3\Psi^2, \Psi^3 - \Psi\Phi - \Omega^3\big) - \Omega}{\wp\big(\tau + K, \Phi + 3\Psi^2, \Psi^3 - \Psi\Phi - \Omega^3\big) - \Psi}\Big), \quad \text{where} \begin{cases} \Psi = \dfrac{2C_2 + 2C_1 + M^2}{6}, \\ \Omega = \dfrac{MC_1}{2}, \\ \Phi = 4C_2H_1 - C_1^2. \end{cases} \tag{4.25}$$

[1]See for an example [69], page 174.

| $x$ | $f(x)$ |
|---|---|
| $+\infty$ | $>0$ |
| $\sqrt{C_2+H_1}$ | $\leq 0$ |
| $p^3(0)$ | $\geq 0$ |
| $-\sqrt{C_2+H_1}$ | $\leq 0$ |
| $-\infty$ | $>0$ |

Table 4.1: The changes in the sign of $f(x)$.

Note that the last two arguments of $\wp$ are not its periods but the elliptic invariants. In order to fit with the given initial condition $p^3(0)$ it is possible to show that one has to choose the value of $K$ according to the relation:

$$\wp\big(K, \Phi+3\Psi^2, \Psi^3-\Psi\Phi-\Omega^3\big) = \frac{(p^3(0))^2+\dot{p}^3|_{t=0}-\Psi}{2}. \tag{4.26}$$

By the expression (4.25) it isn't simple to see that $p^3(\tau)$ is actually bounded. It is however possible to make plain this points by writing the solution in terms of the roots of (4.24). Let me clarify this point. The key observation is that if one look at the r.h.s. of (4.24) as an algebraic equation for $p^3$, so that the equation is a quartic in this variable, then it is possible to show that it has always four *real* roots. This allows to arrange them in order of crescent magnitude and then to infer some properties of the solution of (4.24). The Casimirs and integrals are fixed if one fixes the initial conditions, so one can assume that the dynamical variables in these quantities are specified by their initial values, say at $\tau=0$. With this in mind, by posing $p^3(\tau)=x$, I rewrite (4.24) as:

$$\big(x^2-2C_2\big)\big(x^2-2H_1\big)-\big(Mx-C_1\big)^2 = f(x). \tag{4.27}$$

If one can find five distinct points where $f(x)$ changes its sign, then equation (4.27) has four real roots. Such points are collected in Table 4.1.
Note that iff the initial condition are such that $J^1(0)=J^2(0)=p^1(0)=p^2(0)=0$, then $p^3(0)$ is equal to one of the two points $\pm\sqrt{C_2+H_1}$. But in this case the four roots are $x=p^3(0)$, $x=J^3(0)-p^3(0)$, $x=-J^3(0)-p^3(0)$, with $x=p^3(0)$ a double root, so that also in this case there are four real roots. So in general (4.27) has four real roots, with *at most* three equals and with *at least* one negative (for obvious reasons I do not consider the trivial case when the body is initially at rest). Given the reality of the roots, it is possible now to sort them in an increasing order of magnitude so that by labelling with $a$, $b$, $c$, $d$, one can assume $a\geq b\geq c\geq d$. Note also that $p^3(0)$ lies in the interval $(c,b)$. Equation (4.24) can be written then as:

$$(\dot{p}^3)^2 = (p^3-a)(p^3-b)(p^3-c)(p^3-d). \tag{4.28}$$

The integration of (4.28) is reduced to a standard elliptic integral of the first kind by the substitution [7] $z^2 = \frac{(b-p^3(\tau))(a-c)}{(a-p^3(\tau))(b-c)}$. After some algebra one obtains:

$$p^3(\tau) = \frac{b - \mu^2 a\,\mathrm{sn}(v+\eta\tau,k)^2}{1-\mu^2\,\mathrm{sn}(v+\eta\tau,k)^2}, \qquad \text{where} \qquad \begin{cases} \mu^2 = \dfrac{b-c}{a-c}, \\ k^2 = \dfrac{(a-d)(b-c)}{(a-c)(b-d)}, \\ \eta^2 = \dfrac{(a-c)(b-d)}{4}, \\ v = \mathrm{sn}^{-1}\Big(\sqrt{\dfrac{(b-p^3(0))(a-c)}{(a-p^3(0))(b-c)}}, k\Big). \end{cases} \tag{4.29}$$

As can be seen by the expansion of $p^3(\tau)$ in the neighborhood of $\tau = 0$, the sign of $\eta$ has to be chosen according to the sign of $\dot{p}^3(0)$, i.e. $\mathrm{sgn}(\eta) = -\mathrm{sgn}(\dot{p}^3(0)) = -\mathrm{sgn}\big(J^2(0)p^1(0) - J^1(0)p^2(0)\big)$. Note that $p^3(\tau)$ is bounded in the set $(c,a)$, that is $c \le p^3(\tau) \le a$. Having (4.29) it is a simple matter to write down the expressions for the other dynamical variables:

$$\begin{cases} s(\tau) = \sqrt{2C_2 - (p^3(\tau))^2}, \\ \sigma(\tau) = \sqrt{2H_1 - (p^3(\tau))^2}, \\ \cos(\psi(\tau)) = \dfrac{C_1 - Mp^3(\tau)}{s(\tau)\sigma(\tau)}, \\ f(\tau) = f(0) + \displaystyle\int_0^\tau \Big(\frac{s(z)}{\sigma(z)} p^3(z)\cos(\psi(z)) - \frac{\alpha}{\beta}M\Big)dz. \end{cases} \tag{4.30}$$

## 4.4 Bäcklund transformations

In order to construct the Bäcklund transformations for the dynamical system defined by the Lax matrix (4.10), one needs of an anstaz for the dressing matrix $D(\lambda)$ intertwining the Lax matrices corresponding to two different solutions of the equations of motion. Since the Poisson structure of the Kirchhoff top is shared with that of the Gaudin magnet, one can take the same ansätze: formally the dressing matrix is that of the trigonometric Gaudin magnet (3.96):

$$D(\lambda) = \begin{pmatrix} \sin(\lambda-\lambda_0-\mu) + PQ\,\cos(\lambda-\lambda_0) & P\,\cos(\mu) \\ Q\,\sin(2\mu) - PQ^2\cos(\mu) & \sin(\lambda-\lambda_0+\mu) - PQ\,\cos(\lambda-\lambda_0). \end{pmatrix} \tag{4.31}$$

I recall that $\lambda_0$ and $\mu$ are arbitrary constants (the two Bäcklund parameters) and $P$ and $Q$ are, up to now, indeterminate dynamical variables. The aim is to find an expression for $P$ and $Q$ in terms of only one set of dynamical variables, say the old one, so that the similarity transformation

$$\tilde{L}(\lambda)D(\lambda) = D(\lambda)L(\lambda),$$

allow to write down the explicit map between the two sets of variables. To do this, I use again the spectrality property: the determinant of $D(\lambda)$ is proportional to $\sin(\lambda - \lambda_0 - \mu)\sin(\lambda - \lambda_0 + \mu)$, so it has (up to turns around the unit circle) two zeros, one for $\lambda = \lambda_0 + \mu$ and one for $\lambda = \lambda_0 - \mu$. For these values of $\lambda$, $D(\lambda)$ is clearly a rank one matrix, so it has two one dimensional kernels, one for $\lambda = \lambda_0 + \mu$ and one for $\lambda = \lambda_0 - \mu$. One more time I recall that these kernels happens to be, respectively, also the eigenvectors of $L(\lambda_0 + \mu)$ and $L(\lambda_0 - \mu)$ with eigenvalues given by:

$$\gamma_\pm^2 = A^2(\lambda) + B(\lambda)C(\lambda)\big|_{\lambda=\lambda_0\pm\mu}, \tag{4.32}$$

where $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ are as defined in (4.10). The eigenvectors relations give the links between $P$, $Q$ and the old dynamical variables. The two kernels are given by:

$$|K_+\rangle = \begin{pmatrix} 1 \\ -Q \end{pmatrix}, \qquad |K_-\rangle = \begin{pmatrix} P \\ 2\,\sin(\mu) - PQ \end{pmatrix}, \tag{4.33}$$

and then readily follow the expressions for $Q$ and $P$:

$$Q = Q(\lambda_0 + \mu) = \left.\frac{A(\lambda) - \gamma(\lambda)}{B(\lambda)}\right|_{\lambda=\lambda_0+\mu} \qquad \frac{1}{P} = \frac{Q(\lambda_0+\mu) - Q(\lambda_0-\mu)}{2\,\sin(\mu)}. \tag{4.34}$$

All these results are formally the same as for the trigonometric Gaudin model but, for completeness, again I have reported them here. Taking the residue at the pole in $\lambda = 0$ and the value at $\lambda = \frac{\pi}{2}$ one obtain the explicit maps as below:

$$\begin{aligned}\tilde{p}^- =&\frac{1}{\Delta(\lambda=0)}\Big(\big(\sin(\lambda_1) - PQ\cos(\lambda_0)\big)^2 p^- - P^2\cos(\mu)^2 p^+ + \\ &+ 2P\cos(\mu)\big(\sin(\lambda_1) - PQ\cos(\lambda_0)\big)p^3\Big),\end{aligned} \tag{4.35a}$$

$$\begin{aligned}\tilde{p}^+ =&\frac{1}{\Delta(\lambda=0)}\Big(\big(\sin(\lambda_2) + PQ\cos(\lambda_0)\big)^2 p^+ - Q^2\cos(\mu)^2\big(2\sin(\mu) - PQ\big)^2 p^- + \\ &- 2Q\cos(\mu)\big(2\sin(\mu) - PQ\big)\big(\sin(\lambda_2) + PQ\cos(\lambda_0)\big)p^3\Big),\end{aligned} \tag{4.35b}$$

$$\begin{aligned}\tilde{p}^3 =&\frac{1}{\Delta(\lambda=0)}\Big(2PQ\cos(\mu)^2\big(2\sin(\mu) - PQ\big)p^3 - P\cos(\mu)\big(\sin(\lambda_2) + PQ\cos(\lambda_0)\big)p^+ + \\ &+ Q\cos(\mu)\big(2\sin(\mu) - PQ\big)\big(\sin(\lambda_1) - PQ\cos(\lambda_0)\big)p^-\Big) + p^3,\end{aligned} \tag{4.35c}$$

$$\begin{aligned}\tilde{J}^- =&\frac{1}{\Delta(\lambda=\frac{\pi}{2})}\Big(\big(\cos(\lambda_1)+PQ\sin(\lambda_0)\big)^2 J^- - P^2\cos(\mu)^2 J^+ + \\ &- 2P\cos(\mu)\big(\cos(\lambda_1)+PQ\sin(\lambda_0)\big)p^3\Big),\end{aligned} \tag{4.35d}$$

$$\begin{aligned}\tilde{J}^+ =&\frac{1}{\Delta(\lambda=\frac{\pi}{2})}\Big(\big(\cos(\lambda_2)-PQ\sin(\lambda_0)\big)^2 J^+ - Q^2\cos(\mu)^2\big(2\sin(\mu)-PQ\big)^2 J^- + \\ &+ 2Q\cos(\mu)\big(2\sin(\mu)-PQ\big)\big(\cos(\lambda_2)-PQ\sin(\lambda_2)\big)p^3\Big),\end{aligned} \tag{4.35e}$$

$$\tilde{J}^3 = J^3. \tag{4.35f}$$

where $\Delta(\lambda)$ is the determinant of $D(\lambda)$, given by:

$$\Delta(\lambda) = \sin(\lambda-\lambda_1)\sin(\lambda-\lambda_2)\big(1-2PQ\sin(\mu)+P^2Q^2\big),$$

and, for brevity, I put $\lambda_1$ and $\lambda_2$ for $\lambda_0+\mu$ and $\lambda_0-\mu$ respectively. In the next Section I will show that, provided $\lambda_0 \in \mathbb{R}$ and $\mu \in i\mathbb{R}$, this two-point transformation is actually a time discretization of a one parameter family of continuous flows having the same integrals of motion (4.12), (4.13) as the continuous dynamical system ruled by the physical Hamiltonian (4.14). With the above constraints on the parameters, the transformations become physical, mapping real variables into real variables. Furthermore these transformations are symplectic. The simplecticity of the transformations simply follow from the simplecticity of the maps for the Gaudin magnet, given in subsection 3.2.2.
Next, I will formulate the conjecture that, provided $\lambda_0 \in \mathbb{R}$ and $\mu \in i\mathbb{R}$, this two-point transformation is not only a time discretization of a one parameter family of continuous flows equipped with the same integrals of motion (4.12), (4.13), but it has also the *same orbits* as the continuous dynamical system ruled by the physical Hamiltonian (4.14). The above conjecture will be verified to hold in a couple of special cases, where the explicit solution of the recurrences defined by the maps (4.35) will be also derived, and shown to be interpolated by the solution to the evolution equations for the continuous Kirchhoff top. An example of numerical evidences of the truth of the conjecture will be also reported.

## 4.5 Continuum limit and discrete dynamics

As shown in subsection 3.2.3, to ensure reality of the maps (4.35), one has to require the dressing matrix $D$ to be proportional to an unitary matrix and this holds true iff $\lambda_\pm$ are mutually complex conjugate, i.e. iff $\lambda_0$ is real and $\mu$ is pure imaginary. So I set:

$$\lambda_+ = \lambda_0 + i\frac{\epsilon}{2}, \qquad \lambda_- = \lambda_0 - i\frac{\epsilon}{2}, \tag{4.36}$$

In the limit $\epsilon \to 0$ the relations (4.35) go into the identity map. Indeed $\epsilon$ plays the role of the time step for the one parameter ($\lambda_0$) discrete dynamics defined by the Bäcklund transformations. By following the trigonometric case of the Gaudin magnet, in order to identify the continuous limit of these discrete dynamics I take the Taylor expansion of the dressing matrix at order $\epsilon$, obtaining:

$$D(\lambda) = \sin(\lambda - \lambda_0)\mathbb{1} - \frac{i\epsilon}{2\gamma(\lambda_0)} \begin{pmatrix} A(\lambda_0)\cos(\lambda-\lambda_0) & B(\lambda_0) \\ C(\lambda_0) & -A(\lambda_0)\cos(\lambda-\lambda_0) \end{pmatrix} + O(\epsilon^2), \tag{4.37}$$

where the functions $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ are given by (4.10), and $\gamma(\lambda)^2 = A(\lambda)^2 + B(\lambda)C(\lambda)$. By inserting this expression in the equation defining the Bäcklund transformations

$$\tilde{L}(\lambda)D(\lambda) = D(\lambda)L(\lambda),$$

one arrives at the Lax pair for the continuous flow:

$$\dot{L}(\lambda) = [L(\lambda), M(\lambda, \lambda_0)], \tag{4.38}$$

where the time derivative is defined by $\dot{L} = \lim_{\epsilon\to 0} \frac{\tilde{L}-L}{\epsilon}$.
The matrix $M(\lambda, \lambda_0)$ has the explicit form:

$$M(\lambda, \lambda_0) = \frac{i}{2\gamma(\lambda_0)} \begin{pmatrix} A(\lambda_0)\cot(\lambda-\lambda_0) & \frac{B(\lambda_0)}{\sin(\lambda-\lambda_0)} \\ \frac{C(\lambda_0)}{\sin(\lambda-\lambda 0)} & -A(\lambda_0)\cot(\lambda-\lambda_0) \end{pmatrix}. \tag{4.39}$$

Again the generating function of the integrals of motion is the Hamiltonian for the dynamical system (4.38):

$$\dot{L}_{ij}(\lambda) = \{\gamma(\lambda_0), L_{ij}(\lambda)\}. \tag{4.40}$$

It is clear that the dynamical system given by (4.40) possesses the integrals (4.13), because of (4.11). Moreover there are some evidences, that will be reported in the following, that the continuous and the discrete system share the same orbits too.
First of all I note that the direction of the continuous flow that obtains in the continuum limit from the discrete dynamics defined by the Bäcklund transformations (4.35), and that of the Kirchhoff top (4.3) with the kinetic energy $T$ given by (4.14), can be made parallel. In fact the shape of the orbits are unchanged if one takes an arbitrary $C^1$ function of the Hamiltonian $\gamma(\lambda_0)$ as a new Hamiltonian in (4.40), since this operation amounts just to a time rescaling (for every fixed orbit $\gamma(\lambda_0)$ is constant). Accordingly, I take as Hamiltonian function $\frac{w\gamma(\lambda_0)^2}{2}$, where $w$ is, so far, an arbitrary constant. The expression (4.11) allows to write the explicit equations of motion for a generic function of the dynamical variables $\mathcal{F}(\mathbf{p}, \mathbf{J})$:

$$\dot{\mathcal{F}}(\mathbf{p}, \mathbf{J}) = \{\frac{w\gamma(\lambda_0)^2}{2}, \mathcal{F}(\mathbf{p}, \mathbf{J})\} = \{w\frac{H_1}{\sin(\lambda_0)^2} + w\frac{H_0\cos(\lambda_0)^2}{\sin(\lambda_0)^2}, \mathcal{F}(\mathbf{p}, \mathbf{J})\}. \tag{4.41}$$

This has to be compared with with the equations of motion for the physical Hamiltonian (4.14):

$$\dot{\mathcal{F}}(\mathbf{p},\mathbf{J}) = \{\frac{H_1}{B_1} + \frac{H_0}{B_3}, \mathcal{F}(\mathbf{p},\mathbf{J})\}. \tag{4.42}$$

The two expressions coincide by identifying:

$$w = \frac{1}{B_1} - \frac{1}{B_3}, \qquad \sin(\lambda_0)^2 = \frac{B_3 - B_1}{B_3}. \tag{4.43}$$

In other words, the physical flow is the continuum limit of the discretized one. Now I make the following

**Conjecture**: *For any fixed $\lambda_0$, there exist a re-parametrization of $\epsilon$, $\epsilon \to T_{\lambda_0}$, possibly depending by the integrals and the Casimir functions, such that $T = \epsilon + O(\epsilon^2)$, and at all order in $T$ the continuous orbits of the physical flow interpolate the discrete orbits defined by the Bäcklund transformations, provided that $\lambda_0$ is chosen according to (4.43).*

This is equivalent to say that, for any fixed $\lambda_0$, through the above reparameterization, Bäcklund transformations form a one parameter ($T$) group of transformations, obeying the linear composition law $\mathrm{BT}_{T_1} \circ \mathrm{BT}_{T_2} = \mathrm{BT}_{T_1+T_2}$, "$\circ$" being the composition. Note also that, *if the conjecture is true*, then, since at first order in $\epsilon$ (and therefore in $T$) the flow is ruled by the hamiltonian $\gamma(\lambda_0)$, one has:

$$\tilde{x}^n = e^{nT\{\gamma(\lambda_0),\cdot\}} x = x + nT\{\gamma(\lambda_0), x\} + \frac{n^2T^2}{2}\{\gamma(\lambda_0), \{\gamma(\lambda_0), x\}\} + \ldots,$$

where $\tilde{x}^n$ means the n-th iteration of the Bäcklund transformations with the same parameter $T$ .
In the figures (4.1) and (4.2) I report respectively an example of the orbit for the variables $(p^1(t), p^2(t), p^3(t))$ for the continuous flow ruled by the Hamiltonian (4.14) and of the corresponding discrete flow obtained by iterating the Bäcklund transformations. The initial conditions are the same and the value of $\lambda_0$ has been chosen so to make the continuous limit of the discrete dynamics parallel to the continuous flow of the Kirchhoff top. They overlap exactly. It could be interesting to compare this discretization with, for example, the Runge-Kutta integration methods.
In the next Section, assuming the conjecture to hold true, I will show a way to find the parameter $T$. There, will be given as well analytic results in two particular cases, where the continuous flow is periodic, and not just quasiperiodic. Also if the conjecture is obviously true in these cases, because the motion is one-dimensional, they show how the parameter $T$ can be found.

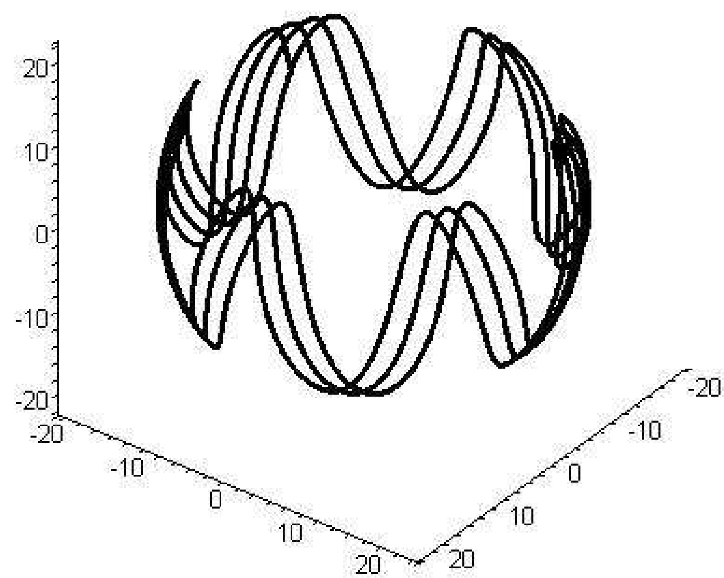


Figure 4.1: initial conditions: $p^1(0) = 15$, $p^2(0) = -12.13$, $p^3(0) = -10$, $J^1(0) = 1$, $J^2(0) = -4$, $J^3(0) = 3$. Moments of inertia: $B_1 = 1$, $B_3 = \sec(0.1)^2$

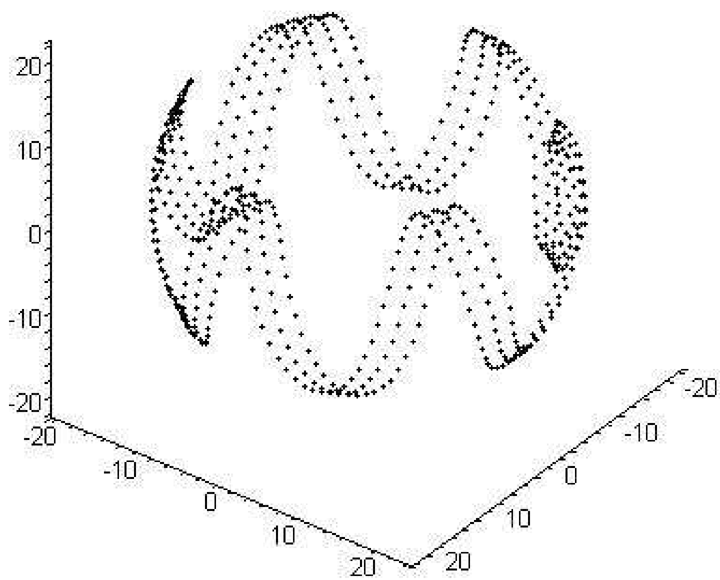


Figure 4.2: input parameters: $p^1(0) = 15$, $p^2(0) = -12.13$, $p^3(0) = -10$, $J^1(0) = 1$, $J^2(0) = -4$, $J^3(0) = 3$, $\lambda_0 = 0.1$, $\epsilon = 0.1$

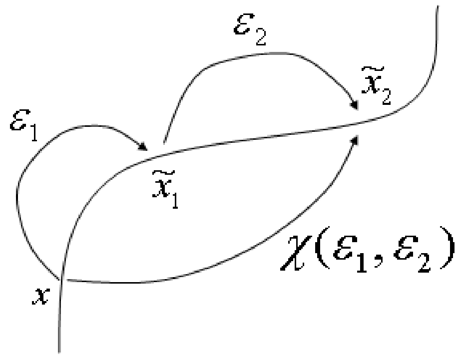


Figure 4.3

### 4.5.1 Integrating the Bäcklund transformations: special examples

Let me assume to have a smooth transformation, that I indicate with $\tilde{x} = f(x, \epsilon)$, where the parameter $\epsilon$ plays the role of the time step, such that $f(x, 0) = x$. By $\tilde{x}^n$ I denote the n-th iteration of the map, so that $\tilde{x}^0 = x$, $\tilde{x}^1 = f(x, \epsilon)$, $\tilde{x}^2 = f(f(x, \epsilon), \epsilon)$ and so on. Solving the Bäcklund map amounts to find $\tilde{x}^n$ as a function of $x$, $n$ and $\epsilon$. Now I will show that, under given assumptions, there is indeed a positive answer to this question. I will follow a simple argument, well known in group theory [48].
Suppose to do a transformation from $x$ to $\tilde{x}^1$ with parameter $\epsilon_1$ and then another one from $\tilde{x}^1$ to $\tilde{x}^2$ with parameter $\epsilon_2$. Suppose also that there exist a parameter $\epsilon_3$ linking directly $x$ to $\tilde{x}^2$. As the Bäcklund transformations are smooth, varying continuously $\epsilon_1$ or $\epsilon_2$ corresponds to a continuous variation in $\epsilon_3$: the Bäcklund transformations define $\epsilon_3$ as a continuous function of $\epsilon_1$ and $\epsilon_2$, say $\epsilon_3 = \chi(\epsilon_1, \epsilon_2)$ (see fig. (4.3)). Now consider infinitesimal transformations: a small change in the parameter $\epsilon$ take the point $\tilde{x}^1$ to a near point $\tilde{x}^1 + d\tilde{x}^1$:

$$\tilde{x}^1 + d\tilde{x}^1 = f(x, \epsilon + d\epsilon). \tag{4.44}$$

But one can arrive at the same point by starting from $\tilde{x}^1$ and acting on it with a transformation near the identity, say with the small parameter $\delta\epsilon$:

$$\tilde{x}^1 + d\tilde{x}^1 = f(\tilde{x}^1, \delta\epsilon). \tag{4.45}$$

The relation between the parameters now reads:

$$\epsilon + d\epsilon = \chi(\epsilon, \delta\epsilon). \tag{4.46}$$

Obviously $\chi(\epsilon, 0) = \epsilon$, so:

$$d\epsilon = \left.\frac{\partial\chi}{\partial\delta\epsilon}\right|_{\delta\epsilon=0} \delta\epsilon \doteq \tau(\epsilon)\delta\epsilon. \tag{4.47}$$

The relation (4.45) tells us that:

$$d\tilde{x}^1 = \left.\frac{\partial f(\tilde{x}^1, \delta\epsilon)}{\partial \delta\epsilon}\right|_{\delta\epsilon=0} \delta\epsilon \doteq \zeta(\tilde{x}^1)\delta\epsilon. \tag{4.48}$$

The last expression together with (4.47) gives:

$$\int_x^{\tilde{x}^1} \frac{dy}{\zeta(y)} = \int_0^\epsilon \frac{d\lambda}{\tau(\lambda)} \doteq T. \tag{4.49}$$

This means that there exists a function, say $V$, such that:

$$V(\tilde{x}^1) = V(x) + T \Longrightarrow V(\tilde{x}^n) = V(x) + nT \tag{4.50}$$

Formally one can write this expression as $\tilde{x}^n = V^{-1}(V(x)+nT)$. However, for $n = 1$ one must have $\tilde{x}^1 = f(x, \epsilon(T))$, yielding $\tilde{x}^n = f(x, \epsilon(nT))$. The continuous flow discretized is simply given by $x(t) = f(x, \epsilon(t))$ where $x$ is the initial condition ($x(t = 0) = x$).
In the following I will present two particular cases, both corresponding to periodic flows, where the Bäcklund transformations can be explicitely integrated.

**Example 1**

Consider the invariant submanifold $\mathbf{p} = (X, 0, Z)$, $\mathbf{J} = (0, Y, 0)$. Since now $H_0 = 0$, the freedom to have a parameter $\lambda_0$ in (4.40) is just a scaling in time, so one can freely fix it: by now I pose $\lambda_0 = \frac{\pi}{2}$. With this choice the interpolating Hamiltonian flow discretized by the maps (4.35) is given simply by $\mathcal{H} = \sqrt{Y^2 + Z^2}$. So, as seen at the beginning of this Section, in order to have real transformations one has to pose $\lambda_1 = \frac{\pi}{2} + i\epsilon$ and $\lambda_2 = \frac{\pi}{2} - i\epsilon$. The Bäcklund transformation can be now conveniently written in terms of a single function $R$ of $\epsilon$, $X$, $Y$ and $Z$:

$$\tilde{X} = \frac{4R\sinh(\epsilon)(R^2+1)}{(R^2-1)^2 + 4\cosh(\epsilon)^2 R^2} Z + \frac{(R^2+1)^2 - 4R^2\sinh(\epsilon)^2}{(R^2-1)^2 + 4R^2\cosh(\epsilon)^2} X, \tag{4.51a}$$

$$\tilde{Y} = \frac{4R\cosh(\epsilon)(R^2-1)}{(R^2-1)^2 + 4\cosh(\epsilon)^2 R^2} Z - \frac{(R^2-1)^2 - 4R^2\cosh(\epsilon)^2}{(R^2-1)^2 + 4R^2\cosh(\epsilon)^2} Y, \tag{4.51b}$$

$$\tilde{Z} = \frac{(R^2+1)^2 - 4R^2\sinh(\epsilon)^2}{(R^2-1)^2 + 4R^2\cosh(\epsilon)^2} Z - \frac{4R\sinh(\epsilon)(R^2+1)}{(R^2-1)^2 + 4\cosh(\epsilon)^2 R^2} X, \tag{4.51c}$$

$$R \doteq \frac{Z - \sqrt{(\mathcal{H}^2\cosh(\epsilon)^2 - 2C_2\sinh(\epsilon)^2)}}{X\sinh(\epsilon) + Y\cosh(\epsilon)}.$$

Note that the two constants under square root in the numerator of $R$ are the Hamiltonian $\mathcal{H} = \sqrt{Y^2 + Z^2}$ and the Casimir function $C_2 = \frac{X^2+Z^2}{2}$. To solve the recurrences

(4.51) one has to find $\epsilon$ as a function of the parameter $T$ defined in (4.49). To this end, first note that $\left.\frac{d\tilde{Z}}{d\epsilon}\right|_{\epsilon=0} = \frac{2XY}{\mathcal{H}}$, so that by the relations (4.49) one has:

$$\int_Z^{\tilde{Z}} \mathcal{H}\frac{d\tilde{Z}}{2\tilde{X}\tilde{Y}} = \int_0^{\epsilon} \mathcal{H}\frac{1}{2\tilde{X}\tilde{Y}}\frac{d\tilde{Z}}{d\epsilon}d\epsilon = \int_0^{\epsilon}\mathcal{H}\frac{d\epsilon}{\sqrt{\mathcal{H}^2\cosh(\epsilon)^2 - 2C_2\sinh(\epsilon)^2}} = T. \quad (4.52)$$

All that one has to do now is to perform the integral, invert the result to find $\epsilon$ as a function of $T$, then plug the result into (4.51) and replace $T$ by $nT$: this gives the solution to the Bäcklund recurrences. After some manipulations with the Jacobian elliptic functions I arrive at the simple result:

$$\cosh(\epsilon) = \frac{1}{\mathrm{cn}(T, \frac{\sqrt{2C_2}}{\mathcal{H}})}, \qquad \sinh(\epsilon) = \frac{\mathrm{sn}(T, \frac{\sqrt{2C_2}}{\mathcal{H}})}{\mathrm{cn}(T, \frac{\sqrt{2C_2}}{\mathcal{H}})}. \quad (4.53)$$

With these positions one can write down the expressions for $\tilde{X}^n$, $\tilde{Y}^n$ and $\tilde{Z}^n$:

$$\tilde{X}^n = \frac{4R\mathrm{sn}(nT)\mathrm{cn}(nT)(R^2+1)}{(R^2-1)^2\mathrm{cn}(nT)^2+4R^2}Z + \frac{(R^2+1)^2\mathrm{cn}(nT)^2 - 4R^2\mathrm{sn}(nT)^2}{(R^2-1)^2\mathrm{cn}(nT)^2+4R^2}X, \quad (4.54a)$$

$$\tilde{Y}^n = \frac{4R\mathrm{cn}(nT)(R^2-1)}{(R^2-1)^2\mathrm{cn}(nT)^2+4R^2}Z - \frac{(R^2-1)^2\mathrm{cn}(nT)^2 - 4R^2}{(R^2-1)^2\mathrm{cn}(nT)^2+4R^2}Y, \quad (4.54b)$$

$$\tilde{Z}^n = \frac{(R^2+1)^2\mathrm{cn}(nT)^2 - 4R^2\mathrm{sn}(nT)^2}{(R^2-1)^2\mathrm{cn}(nT)^2+4R^2}Z - \frac{4R\mathrm{sn}(nT)\mathrm{cn}(nT)(R^2+1)}{(R^2-1)^2\mathrm{cn}(nT)^2+4R^2}X, \quad (4.54c)$$

$$R = \frac{Z\mathrm{cn}(nT) - \sqrt{(\mathcal{H}^2 - 2C_2\mathrm{sn}(nT)^2)}}{X\mathrm{sn}(nT)+Y},$$

where for brevity I have omitted the elliptic modulus $\frac{\sqrt{2C_2}}{\mathcal{H}}$ in the Jacobian elliptic functions "sn" and "cn". Note that, posing in (4.52) $2T = t$, that is

$$\cosh(\epsilon) = \frac{1}{\mathrm{cn}(\frac{t}{2}, \frac{\sqrt{2C_2}}{\mathcal{H}})}, \qquad \sinh(\epsilon) = \frac{\mathrm{sn}(\frac{t}{2}, \frac{\sqrt{2C_2}}{\mathcal{H}})}{\mathrm{cn}(\frac{t}{2}, \frac{\sqrt{2C_2}}{\mathcal{H}})}, \quad (4.55)$$

in (4.51), then one has the *general solution* of the dynamical system ruled by the interpolating Hamiltonian flow $\mathcal{H} = \sqrt{Z^2+Y^2}$, that is the value that takes the hamiltonian $\gamma(\lambda_0)$ (4.11) on the invariant submanifold considered in this example for $\lambda_0$=$\frac{\pi}{2}$. The equations of motion are given by $\mathcal{H}\dot{X} = -YZ$, $\mathcal{H}\dot{Y} = -XZ$, $\mathcal{H}\dot{Z} = XY$.
Obviously this general solution coincide with that found by a direct integration of the previous equation of motion, i.e. with $Z = \sqrt{2C_2}\,\mathrm{sn}(t+v)$, $X = \sqrt{2C_2}\,\mathrm{cn}(t+v)$ and $Y = \mathcal{H}\mathrm{dn}(t+v)$, where the elliptic modulus of this functions is again $\frac{\sqrt{2C_2}}{\mathcal{H}}$ and where $v$ is such that $\mathrm{sn}(v) = \frac{Z}{\sqrt{2C_2}}$.

**Example 2**

In the next example I consider the invariant submanifold $\mathbf{p} = (x, y, 0)$, $\mathbf{J} = (0, 0, z)$. Again, in order to have real transformations I pose $\lambda_1 = \lambda_0 + i\epsilon$ and $\lambda_2 = \lambda_0 - i\epsilon$ with $\lambda_0$ and $\epsilon$ real. In terms of $p^{\pm} = x \pm i\, y$, the maps (4.35) become:

$$\begin{aligned}
\tilde{p}^- &= \frac{z\sin(\lambda_0 - i\epsilon) - \sqrt{z^2\sin(\lambda_0 - i\epsilon)^2 + p^-p^+}}{z\sin(\lambda_0 + i\epsilon) - \sqrt{z^2\sin(\lambda_0 + i\epsilon)^2 + p^-p^+}} p^-,\\
\tilde{p}^+ &= \frac{z\sin(\lambda_0 + i\epsilon) - \sqrt{z^2\sin(\lambda_0 + i\epsilon)^2 + p^-p^+}}{z\sin(\lambda_0 - i\epsilon) - \sqrt{z^2\sin(\lambda_0 - i\epsilon)^2 + p^-p^+}} p^+,\\
\tilde{z} &= z.
\end{aligned} \tag{4.56}$$

To find the relation defining the parameter $T$ (4.49), first one has to find the expression of $\left.\frac{d\tilde{p}^-}{d\epsilon}\right|_{\epsilon=0}$:

$$\left.\frac{d\tilde{p}^-}{d\epsilon}\right|_{\epsilon=0} = \frac{2iz\cos(\lambda_0)p^-}{\sqrt{z^2\sin(\lambda_0)^2 + p^+p^-}};$$

then by using (4.49) one has:

$$\begin{aligned}
&\frac{\sqrt{z^2\sin(\lambda_0)^2 + p^+p^-}}{2\cos(\lambda_0)} \int_{p^-}^{\tilde{p}^-} \frac{d\tilde{p}^-}{iz\tilde{p}^-} = T = \\
&= \frac{\sqrt{z^2\sin(\lambda_0)^2 + p^+p^-}}{2\cos(\lambda_0)} \int_0^{\epsilon} \left( \frac{\cos(\lambda_0 + i\epsilon)}{\sqrt{z^2\sin(\lambda_0 + i\epsilon)^2 + p^+p^-}} + \frac{\cos(\lambda_0 - i\epsilon)}{\sqrt{z^2\sin(\lambda_0 - i\epsilon)^2 + p^+p^-}} \right) d\,\epsilon,
\end{aligned} \tag{4.57}$$

or more explicitely:

$$\begin{aligned}
&\frac{2\,i\cos(\lambda_0)T}{\sqrt{z^2\sin(\lambda_0)^2 + p^+p^-}} = \operatorname{arcsinh}\left(\frac{z}{\sqrt{p^+p^-}}\sin(\lambda_0 + i\,\epsilon)\right) - \operatorname{arcsinh}\left(\frac{z}{\sqrt{p^+p^-}}\sin(\lambda_0 - i\,\epsilon)\right) = \\
&= \ln\left(\frac{z\sin(\lambda_0 - i\epsilon) - \sqrt{z^2\sin(\lambda_0 - i\epsilon)^2 + p^-p^+}}{z\sin(\lambda_0 + i\epsilon) - \sqrt{z^2\sin(\lambda_0 + i\epsilon)^2 + p^-p^+}}\right).
\end{aligned} \tag{4.58}$$

The Bäcklund transformations (4.56) now take the simple form:

$$\begin{aligned}
\tilde{p}^- &= \exp\left(\frac{2\,i\cos(\lambda_0)zT}{\sqrt{z^2\sin(\lambda_0)^2 + p^+p^-}}\right) p^-,\\
\tilde{p}^+ &= \exp\left(-\frac{2\,i\cos(\lambda_0)zT}{\sqrt{z^2\sin(\lambda_0)^2 + p^+p^-}}\right) p^+,\\
\tilde{z} &= z.
\end{aligned} \tag{4.59}$$

so that again, as expected, the n-th iteration of the maps $(\tilde{p}^-)^n, (\tilde{p}^+)^n, \tilde{z}^n$ is found by substituting $T$ with $nT$. By posing $2T = t$ in the previous expressions and returning to the real variables $x = \frac{p^+ + p^-}{2}$ and $y = \frac{p^+ - p^-}{2\,i}$, one has the continuous flow:

$$x(t) = x\cos\left(\frac{\cos(\lambda_0)z}{\sqrt{z^2\sin(\lambda_0)^2 + p^+p^-}}t\right) + y\sin\left(\frac{\cos(\lambda_0)z}{\sqrt{z^2\sin(\lambda_0)^2 + p^+p^-}}t\right),$$
$$y(t) = y\cos\left(\frac{\cos(\lambda_0)z}{\sqrt{z^2\sin(\lambda_0)^2 + p^+p^-}}t\right) - x\sin\left(\frac{\cos(\lambda_0)z}{\sqrt{z^2\sin(\lambda_0)^2 + p^+p^-}}t\right),$$

corresponding to the general solution of the continuous system ruled by the value that takes the hamiltonian $\gamma(\lambda_0)$ (4.11) on the invariant submanifold $\mathbf{p} = (x, y, 0)$, $\mathbf{J} = (0, 0, z)$ considered in this example.

# Conclusions

The original objective of the thesis was to continue, and possibly to conclude, a wide research work on Gaudin models started more than ten years ago with the paper of Hone, Kuznetsov and Ragnisco [52] on one hand and those of Sklyanin [110], [111] on the other hand; in these papers there were presented the Bäcklund transformations for the rational Gaudin model in the framework of the research program started in the same years by Sklyanin and Kuznetsov on the applications and properties of such transformations to finite dimensional integrable systems. The analogue generalizations in the trigonometric and elliptic cases were, so far, an open question. Obviously it wasn't only for a matter of shallow obstinacy to generalize an existing construction that I ventured in this puzzle. By a physical point of view, yet the applications to the BCS theory of superconductivity and the related properties of small metallic grains or the connections with the pairing models in nuclear physics legitimate the interest in the Gaudin systems. In the field of classical mechanics, the direct relations, through the procedure of Inönü Wigner contraction and pole coalescence on Lax matrix, with known integrable systems of physical interest, such as the Kirchhoff model describing the motion of a rigid body in an ideal fluid, or the Lagrange top, provide a direct link between the theory of Bäcklund transformations in the framework of finite dimensional systems and the material world. But the full potentialities of the Bäcklund transformations in the applications to practical questions are far from being bounded to particular problems: I hope that by this work has emerged clearly the role played by the Bäcklund transformations as a tool to obtain not only an exact time discretization of the underlying continuous integrable model, but also, as in the explicit examples given in the subsection 4.5.1, the *general solution* of the equations of motion. In this contest the observations made in the section appear to be a novelty, at least to my knowledge. If the original objective has been accomplished, an amount of work remains to do; first of all I have to mention the lacks in the constructions presented in the thesis. Unlike the rational case, I haven't been able to write in any closed form the generating function of the trigonometric and elliptic maps. Does not even help to have the generating function of the isotropic maps, since the two-parameters Bäcklund transformations in this case can be written as the composition of two simpler one-parameter transformations: the same property holds true for the generating functions; indeed in the section 3.1 it is given only the formula for the one parameter generating function, composing in the proper way two such functions one can obtain the expression for the two parameter generating function. Yet in the trigonometric case a factorization of the dressing matrix cannot lead to a one parameter dressing matrix preserving all the symmetries of the problem (cf. with [94]). There are two possibilities: or one is able to find directly the two parameters generating function, or one should look for a symmetry-violating generating function such that their composition restores the symmetry. However, though these details are important by a quantum point of view for their potential connections with the Baxter's $Q$ operator, a series of questions can be posed, for example for integrable infinite dimensional systems, in connection with the solvability of the evolution equations through the Bäcklund transformations.

A feasible perspective in the immediate future could be the construction of the Bäcklund transformations for the algebraic extension of the two-site elliptic Gaudin model: in this case, as shown for example in [86], one obtains a relaxation of the constraints on the arbitrary constants appearing in the Kirchhoff top as given in the chapter 4; other than an obvious generalization of the model studied in that chapter, there may be the possibility to better understand and describe how the Bäcklund transformations act on the Liouville torus of the integrable system.

As regards the result obtained in the thesis, I have constructed, in a general and systematic way, the Bäcklund transformations for the trigonometric and elliptic Gaudin models. I have used the adjective "general" because the composition of enough maps leads to an $N$-parameters Bäcklund transformations, corresponding to a collection of shift on the angle coordinates of the $N$-dimensional Liouville torus of the system: one has to remember that the Bäcklund transformations are indeed canonical maps that preserve the integrals, so their action, as shown by Veselov [127], is precisely a shift of the angle coordinates on the torus (see also the section 1.5.2). Having a cover of the torus, any other canonical transformations can be expressed, at least in principle, in terms of these Bäcklund transformations. The adjective "systematic" refers to the *modus operandi* and the methods employed in the thesis: following the approach of Kuznetsov and Sklyanin [64], I looked at the Bäcklund transformations for the trigonometric and elliptic Gaudin models as canonical transformations; this Hamiltonian point of view has then been enriched showing the remarkable properties of the maps, such as explicitness, symplecticity, spectrality, limits to continuous flows, preservation of the integrals that the maps discretize and the mapping of real solutions to the equations of motion into real solutions. Finally, as an application of the construction found in the trigonometric case, I gave the Bäcklund transformations for the Kirchhoff top describing the motion of a solid in an infinite incompressible fluid: other than the construction itself, it is remarkable in this application the example on the explicit *integration* of the Bäcklund transformations that led to the *general solution* of the equations of motion.

# Appendix A

# Elliptic function formulae

Let $w_1, w_2$ be complex numbers such that their ratio is not real and consider the lattice $\Lambda$ generated by these numbers:

$$\Lambda = \{w \in \mathbb{C} : w = n_1 w_1 + n_2 w_2, n_1, n_2 \in \mathbb{Z}^2\}.$$

The Weierstraß zeta function is given by [7]:

$$\zeta(u) = \frac{1}{u} + \sum_{w \neq 0} \left( \frac{1}{u-w} + \frac{1}{w} + \frac{u}{w^2} \right). \tag{A.1}$$

The Weierstraß $\wp$ function is minus the derivative of $\zeta$:

$$\wp(u) = -\zeta'(u) = \frac{1}{u^2} + \sum_{w \neq 0} \left( \frac{1}{(u-w)^2} - \frac{1}{w^2} \right). \tag{A.2}$$

By denoting the period $w_3$ such that $w_1 + w_2 + w_3 = 0$ and defining the set $e_i$, $i = 1...3$ by $e_i = \wp(\frac{w_i}{2})$, then holds the relation [7]:

$$\wp(u\varpi) = e_2 + \frac{1}{\varpi^2 \mathrm{sn}(u,k)}, \tag{A.3}$$

where $\varpi = (e_1 - e_2)^{-\frac{1}{2}}$ and the elliptic modulus for the Jacobi "sn" function is given by $k^2 = \varpi^2(e_3 - e_2)$. The Jacobi elliptic functions $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$ and $\mathrm{dn}(u,k)$ satisfy the following quasi-periodic relations [7]:

$$\begin{aligned}
\mathrm{sn}(u + 2mK + 2inK', k) &= (-1)^m \mathrm{sn}(u,k), \\
\mathrm{cn}(u + 2mK + 2inK', k) &= (-1)^{m+n} \mathrm{cn}(u,k), \\
\mathrm{dn}(u + 2mK + 2inK', k) &= (-1)^n \mathrm{dn}(u,k),
\end{aligned} \tag{A.4}$$

where $K$ and $K'$ are respectively the complete elliptic integral of the first kind and the complementary integral:

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \qquad K'(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-(1-k^2)t^2)}}. \tag{A.5}$$

The following formulae are useful in proving (3.141):

$$\mathrm{cn}(x \pm y) = \mathrm{cn}(x)\mathrm{cn}(y) \mp \mathrm{dn}(x \pm y)\mathrm{sn}(x)\mathrm{sn}(y), \tag{A.6}$$

$$\mathrm{dn}(x \pm y) = \mathrm{dn}(x)\mathrm{dn}(y) \mp k^2\mathrm{cn}(x \pm y)\mathrm{sn}(x)\mathrm{sn}(y), \tag{A.7}$$

$$\begin{cases} \varpi\left(\zeta(\varpi x) - \zeta(\varpi y)\right) - \frac{\mathrm{sn}(y-x)}{\mathrm{sn}(x)\mathrm{sn}(y)} = a(y-x), \\ a(x) \doteq \varpi\left(\zeta(\varpi x) - \zeta(2\varpi x)\right) - \frac{1}{\mathrm{sn}(2x)}. \end{cases} \tag{A.8}$$

Equations (A.6) and (A.7) are just a consequence of the addition formulae for the Jacobi elliptic functions, (A.8) can be proved in few lines. In fact, suppose that $x$ and $y$ vary while $y-x$ remains constant and equal to $b$. Differentiating $f(x) = \varpi\left(\zeta(\varpi x) - \zeta(\varpi(x+b))\right) - \frac{\mathrm{sn}(b)}{\mathrm{sn}(x)\mathrm{sn}(x+b)}$ with respect to $x$, we see that this function is independent of $x$. Indeed

$$f'(x) = \varpi^2\big(\wp((x+b)\varpi) - \wp(x\varpi)\big) - \frac{\mathrm{sn}(b)\big(\mathrm{sn}(x)\mathrm{sn}(x+b)\big)'}{\big(\mathrm{sn}(x)\mathrm{sn}(x+b)\big)^2}.$$

By using the relation (A.3) and once again the addition formulas for the Jacobi elliptic functions, it is readily shown that $f'(x) = 0$, so $f(x)$ is a constant, that we can take as a function of $b$. This implies the relation $\varpi\left(\zeta(\varpi x) - \zeta(\varpi y)\right) - \frac{\mathrm{sn}(y-x)}{\mathrm{sn}(x)\mathrm{sn}(y)} = a(y-x)$. By posing $y = 2x$ in this equation, we obtain the function $a(x)$ as in (A.8).
Another set of useful formulae, for example for the explicit calculation of the quadratic Poisson bracket (2.14), are given by:

$$\frac{\mathrm{dn}(y)\mathrm{dn}(x-y) - \mathrm{dn}(x)}{\mathrm{sn}(x)\mathrm{sn}(y)\mathrm{sn}(x-y)} = k^2\frac{\mathrm{cn}(x)}{\mathrm{sn}(x)}, \tag{A.9}$$

$$\frac{\mathrm{cn}(y)\mathrm{dn}(x)\mathrm{cn}(x-y) - \mathrm{cn}(x)\mathrm{dn}(y)\mathrm{dn}(x-y)}{\mathrm{sn}(x)\mathrm{sn}(y)\mathrm{sn}(x-y)} = \frac{1-k^2}{\mathrm{sn}(x)}, \tag{A.10}$$

$$\frac{\mathrm{cn}(y)\mathrm{cn}(x-y) - \mathrm{cn}(x)}{\mathrm{sn}(x)\mathrm{sn}(y)\mathrm{sn}(x-y)} = \frac{\mathrm{dn}(x)}{\mathrm{sn}(x)}, \tag{A.11}$$

$$\frac{\mathrm{cn}(y)\mathrm{sn}(x) - \mathrm{sn}(y)\mathrm{cn}(x)\mathrm{dn}(x-y)}{\mathrm{sn}(x-y)} = \mathrm{dn}(y), \tag{A.12}$$

$$\frac{\mathrm{sn}(x)\mathrm{dn}(x-y) - \mathrm{cn}(x-y)\mathrm{sn}(y)}{\mathrm{sn}(x-y)} = \mathrm{dn}(x)\mathrm{cn}(y), \tag{A.13}$$

$$\frac{\mathrm{cn}(x-y)\mathrm{dn}(y)\mathrm{sn}(x)-\mathrm{dn}(x)\mathrm{sn}(y)}{\mathrm{sn}(x-y)}=\mathrm{cn}(x). \tag{A.14}$$

Now let me consider more closely the formula for the generating function of the integrals of the elliptic Gaudin model (3.141). For brevity I pose in the following $v_i=\lambda-\lambda_i$ and $v_{ij}=\lambda_i-\lambda_j$. From (3.136) one has:

$$\begin{aligned}-\det(L(\lambda))=&\sum_{i,j}\frac{\mathrm{cn}(v_i)\mathrm{cn}(v_j)s_i^3s_j^3+s_i^1s_j^1+\mathrm{dn}(v_i)\mathrm{dn}(v_j)s_i^2s_j^2}{\mathrm{sn}(v_i)\mathrm{sn}(v_j)}=\\ &=\sum_{\substack{i,j\\ i\neq j}}\frac{\mathrm{cn}(v_i)\mathrm{cn}(v_j)s_i^3s_j^3+s_i^1s_j^1+\mathrm{dn}(v_i)\mathrm{dn}(v_j)s_i^2s_j^2}{\mathrm{sn}(v_i)\mathrm{sn}(v_j)}+\\ &+\sum_i\frac{c_i^2}{\mathrm{sn}(v_i)^2}-\sum_i\left((s_i^3)^2+k^2(s_i^2)^2\right).\end{aligned} \tag{A.15}$$

By adding and subtracting the quantities $\sum_{i\neq j}\left(\mathrm{dn}(v_{ij})s_i^3s_j^3+k^2\mathrm{cn}(v_{ij})s_i^2s_j^2\right)$ in the last equation and using (A.6) and (A.7), one finds:

$$\begin{aligned}-\det(L(\lambda))=&\sum_i\frac{c_i^2}{\mathrm{sn}(v_i)^2}-\sum_{i,j}\left(s_i^3s_j^3\mathrm{dn}(v_{ij})+k^2s_i^2s_j^2\mathrm{cn}(v_{ij})\right)+\\ &+\sum_{\substack{i,j\\ i\neq j}}\frac{\mathrm{cn}(v_{ij})s_i^3s_j^3+s_i^1s_j^1+\mathrm{dn}(v_{ij})s_i^2s_j^2}{\mathrm{sn}(v_i)\mathrm{sn}(v_j)}.\end{aligned} \tag{A.16}$$

Now, using formula (A.8) on the denominator of the last sum of equation (A.16) and defining

$$H_i=\sum_{j\neq i}^{N}\frac{s_i^3s_j^3\mathrm{cn}(v_{ij})+s_i^2s_k^2\mathrm{dn}(v_{ij})+s_i^1s_k^1}{\mathrm{sn}(v_{ij})},$$

one reaches the result (3.141).

# Appendix B

# The $r$-matrix formalism

In this appendix I give some remarks on Lax matrix structure and $r$-matrix formalism for finite dimensional Hamiltonian systems. For a more detailed account the reader can see [9], [35], [109], [115].

A Lax pair is a couple of matrices, usually denoted as $L$ and $M$, whose elements are functions of the phase space variables and that can depend also by a complex parameter, say $\lambda$, known as the *spectral parameter*. The equations of motion can be rewritten in terms of these two matrices as:

$$\frac{dL}{dt} = [L, M]. \tag{B.1}$$

If one is able to rewrite the evolution equations as a Lax pair, then readily obtains a set of integrals of the system. In fact, by a direct substitution, it is possible to verify that the solution of the equation (B.1) is given by:

$$L(t) = g^{-1}(t)L(0)g(t), \qquad M = g^{-1}\frac{dg}{dt},$$

so that every function of the eigenvalues of $L$ (or equivalently every functions of the trace of $L^n$ for every $n$) is a constant of motion. If the Lax representation provides, for a system with $N$ degrees of freedom, a set of $N$ independent involutive integrals, then the system is integrable in the sense of Liouville [8]. Note that, as shown in [9], there exist a Lax pair for every integrable system. Often in the calculations one needs the Poisson bracket between functions of the elements of the Lax matrix. These elements can be functions of several dynamical variables, so that could be a formidable task to obtain a reasonable result. One of the technique that can be used is to appeal to the $r$-matrix formalism. Actually $r$-matrices have a very deep theoretical meaning, as they *define* the Lie Poisson structure of the dynamical problem. However for the beginner it is easily to take the first of view. An important theorem due to Babelon and Viallet [10] (see also [9]) says that the involution property of the eigenvalues of the

Lax matrix ensure the existence of an $r$-matrix; before to go beyond, let me explain some notations. The point is to calculate the Poisson bracket between the elements of the Lax matrix. By taking a basis $e_{ij}$ for the $N \times N$ matrices, one has:

$$L = \sum_{ij} L_{ij} e_{ij},$$

so that $\{L_{ij}, L_{km}\}$ can be seen as the elements of an $N^2 \times N^2$ matrix; more formally:

$$\{L_1, L_2\} = \sum_{ij,kl} \{L_{ij}, L_{kl}\} e_{ij} \otimes e_{kl},$$

where $L_1 = L \otimes 1\!\!1 = \sum_{ij} L_{ij} \left(e_{ij} \otimes 1\!\!1\right)$ and $L_2 = 1\!\!1 \otimes L = \sum_{ij} L_{ij} 1\!\!1 \otimes (e_{ij})$.
Now, according to Babelon and Viallet, there exist a matrix $r_{12}$, whose elements can depend on the dynamical variables, such that:

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]. \tag{B.2}$$

The notation $r_{12}$ and $r_{21}$ obviously stands for $r_{12} = \sum_{ij,kl} r_{ij,kl} e_{ij} \otimes e_{kl}$ and $r_{21} = \sum_{ij,kl} r_{ij,kl} e_{kl} \otimes e_{ij}$. The matrix $M$ in (B.1) determines the flow of the dynamical variables associated to the Lax matrix, the $r$-matrix determines the Poisson structure at which these variables must obey, so it isn't difficult to think that they have to be related. This is the case. By following [9], one has:

$$\{L_1^n, L_2^m\} = [a_{12}^{nm}, L_1] - [b_{12}^{nm}, L_2] \qquad \begin{cases} a_{12}^{nm} = \displaystyle\sum_{p=0}^{n-1} \sum_{q=0}^{m-1} L_1^{n-p-1} L_2^{m-q-1} r_{12} L_1^p L_2^q, \\ b_{12}^{nm} = \displaystyle\sum_{p=0}^{n-1} \sum_{q=0}^{m-1} L_1^{n-p-1} L_2^{m-q-1} r_{21} L_1^p L_2^q. \end{cases} \tag{B.3}$$

This expression is only a straightforward application of the derivative properties of Poisson brackets. The traces of $L^n$ are conserved quantities; take these quantities as Hamiltonian function, $H_n = Tr(L^n)$. Set $m = 1$ in (B.3) and take the trace over the first space, that labeled by the 1 subscript. Then:

$$\{H_n, L\} = \frac{dL}{dt_n} = [L, nTr_1 \left(L_1^{n-1} r_{21}\right)],$$

so that the matrix $M$ is given by $M = nTr_1 \left(L_1^{n-1} r_{21}\right)$. Note that, for finite dimensional integrable systems that possess a known Lax representation, almost all have a spectral parameter. In this case the eigenvalues of $L$ provide a family of conserved quantities. If relation (B.2) must hold, then also the $r$-matrix must depend on the spectral parameter. The proper generalization of (B.2) reads [9], [109]:

$$\{L_1(\lambda), L_2(\mu)\} = [r_{12}(\lambda, \mu), L_1(\lambda)] - [r_{21}(\mu, \lambda), L_2(\mu)]. \tag{B.4}$$

The Jacobi identity for the Poisson bracket requires some relation for the $r$-matrix to be fulfilled. As a matter of fact, the most extensively studied case is when the $r$-matrix does not depend on dynamical variables, so that it contains only numbers. In this case it has to satisfy the *classical Yang-Baxter equation* [9], [61], [109]:

$$[r_{12}(\lambda,\mu), r_{13}(\lambda,v) + r_{23}(\mu,v)] - [r_{13}(\lambda,v), r_{32}(v,\mu)] = 0. \tag{B.5}$$

In this contest, very important results on the classification of $r$-matrices labeled by Lie algebras have been obtained by Belavin and Drinfeld [14], [15], [16] for unitary $r$-matrices, satisfying the relation

$$r_{12}(\lambda,\mu) = -r_{21}(\mu,\lambda)$$

and depending only on the difference of the spectral parameters, that is $r_{12}(\lambda,\mu) = r_{12}(\lambda-\mu)$. For such matrices, the relation (B.4) takes the simpler form:

$$\{L_1(\lambda), L_2(\mu)\} = [r(\lambda-\mu), L_1(\lambda) + L_2(\mu)], \tag{B.6}$$

with $r \doteq r_{12}$. I will give only a brief account on these results, the reader is referred to [14], [15], [16], [97] for a more detailed account. If the Lax matrix takes values in some Lie algebra $\mathfrak{g}$, the $r$-matrix take values in $\mathfrak{g}\otimes\mathfrak{g}$. Let $X^\alpha$, $\alpha = 1\ldots\dim\mathfrak{g}$ the elements of a basis of $\mathfrak{g}$ with structure constants given by the set $\{c^{\alpha\beta}_\gamma\}$:

$$[X^\alpha, X^\beta] = c^{\alpha\beta}_\gamma X^\gamma.$$

Here and after a summation over repeated index is understood. Belavin and Drinfeld then show [80] that the solution $r(\lambda)$ of the classical Yang-Baxter equation is a meromorphic function with a pole of first order at $\lambda = 0$ with residue given by:

$$\mathrm{res}|_{\lambda=0}\, r(\lambda) = g_{\alpha\beta} X^\alpha X^\beta,$$

where $g_{\alpha\beta}$ is the metric associated to the basis $\{X^\alpha\}_{\alpha=1}^{\dim\mathfrak{g}}$ [48]: $g_{\alpha\beta} = c^j_{\alpha i} c^i_{\beta j}$. The remarkable result can be stated in the following classification: for finite dimensional simple Lie algebras, the only non degenerate solutions to the classical Yang Baxter equation have rational, trigonometric or elliptic dependence on the spectral parameter and are given by:

- rational solution:
$$r(\lambda) = \frac{g_{\alpha\beta} X^\alpha \otimes X^\beta}{\lambda}$$
;
- trigonometric solution:
$$r(\lambda) = \sum_{n=-\infty}^{\infty} \frac{(A^n \otimes \mathbb{1})\, g_{\alpha\beta} X^\alpha \otimes X^\beta}{\lambda - nw}$$
;

- elliptic solution:

$$r(\lambda) = \sum_{n,m=-\infty}^{\infty} \frac{(A^n B^m \otimes \mathbb{1})\, g_{\alpha\beta} X^\alpha \otimes X^\beta}{\lambda - n w_1 - m w_2}, \qquad [A,B] = 0$$

,

where $A$ and $B$ are two finite order commuting automorphism of $\mathfrak{g}$ not having a common fixed point. Note that all solutions can be written as $r(\lambda) = g_{\alpha\beta} X^\alpha \otimes X^\beta f^\alpha(\lambda)$ for some functions $f^\alpha(\lambda)$. The Yang-Baxter equation (B.5) then is equivalent to the following system of functional equations on $\{f^\alpha(\lambda)\}_{\alpha=1}^{\dim\mathfrak{g}}$ [80]:

$$\sum_{\beta,\delta}^{\dim\mathfrak{g}} g_{\alpha\beta} g_{\gamma\delta} c_\eta^{\beta\delta} \left( f^\alpha(\lambda) f^\gamma(\mu) + f^\delta(\lambda-\mu) f^\beta(\lambda) - f^\alpha(\lambda-\mu) f^\delta(\mu) \right) = 0, \tag{B.7}$$

for $\alpha, \gamma, \eta = 1 \ldots \dim\mathfrak{g}$. In the case of $\mathfrak{su}(2)$ algebra, and so for $\mathfrak{su}(2)$ Gaudin models, the explicit form of the functions $f^\alpha$ is [80]:

$$f^1(\lambda) = \begin{cases} \dfrac{1}{\lambda}, & \text{rational case} \\ \dfrac{1}{\sin(\lambda)}, & \text{trigonometric case} \\ \dfrac{1}{\mathrm{sn}(\lambda)}, & \text{elliptic case;} \end{cases} \tag{B.8}$$

$$f^2(\lambda) = \begin{cases} \dfrac{1}{\lambda}, & \text{rational case} \\ \dfrac{1}{\sin(\lambda)}, & \text{trigonometric case} \\ \dfrac{\mathrm{dn}(\lambda)}{\mathrm{sn}(\lambda)}, & \text{elliptic case;} \end{cases} \tag{B.9}$$

$$f^3(\lambda) = \begin{cases} \dfrac{1}{\lambda}, & \text{rational case} \\ \dfrac{\cos(\lambda)}{\sin(\lambda)}, & \text{trigonometric case} \\ \dfrac{\mathrm{cn}(\lambda)}{\mathrm{sn}(\lambda)}, & \text{elliptic case.} \end{cases} \tag{B.10}$$

As noted by Sklyanin [109], to a unitary $r$-matrix it is also possible to associate a *quadratic* Poisson algebra, given by:

$$\{L_1(\lambda), L_2(\mu)\} = [r(\lambda-\mu), L_1(\lambda) L_2(\mu)]. \tag{B.11}$$

Note however that it is always possible to introduce a dynamic $r$-matrix, given by $2\tilde{r}(\lambda - \mu) = r(\lambda - \mu)L_2(\mu) + L_2(\mu)r(\lambda - \mu)$ so that (B.11) reduces to (B.6). A last remark on the links between linear and quadratic $r$-matrix structures: by a formal expansions on some parameter $\epsilon$: $L = 1 + \epsilon L^1 + O(\epsilon^2)$ and $r = \epsilon r^1 + O(\epsilon^2)$, expression (B.11) goes into (B.6) in the limit $\epsilon \to 0$. This limiting procedure has been indeed used in 2.3 to obtain the the Gaudin models as a limit of the lattice Heisenberg models.

# Appendix C

# The Kirchhoff equations

**Preliminary considerations.** In this appendix, following [59], [70], [76], I will review the derivation of the equations of motion of a solid in an infinite, incompressible fluid under no forces.
Suppose to have a fluid in motion subjected to an external impulse per unit mass $\mathbf{I}$ and to an impulsive pressure $\varpi$. Consider the part of the fluid within a closed surface $S$ and equate the impulse to the change of the momentum of the fluid. If $\mathbf{q}'$ and $\mathbf{q}$ are respectively the velocity after and before the application of impulses, then:

$$-\int \varpi \mathbf{n} dS + \int \rho \mathbf{I} dV = \int \rho(\mathbf{q}' - \mathbf{q}) dV.$$

Note that the minus in front of the surface integral is due to the fact that $\mathbf{n}$ is the versor outwards the surface, while the direction of the impulsive pressure is inwards the surface. The volume integral is obviously on the volume enclosed by the surface $S$, $\rho$ is the density of the fluid. Using the Gauss' theorem, one has $\int \varpi \mathbf{n} dS = \int (\mathrm{grad}\, \varpi) dV$, so that, by the arbitrariness of the surface taken, the equation for the impulsive motion follows:

$$\mathbf{I} - \frac{1}{\rho} \mathrm{grad}\, \varpi = \mathbf{q}' - \mathbf{q}. \tag{C.1}$$

Note that, if there aren't external forces and if the motion is irrotational and generated from the rest, that is if $\mathbf{I} = 0$, $\mathbf{q} = 0$ and $\mathbf{q}' = -\mathrm{grad}\, \phi$ for some scalar function $\phi$, known as the velocity potential, then equation (C.1) reduces to $\mathrm{grad}\, \varpi = \rho\, \mathrm{grad}\, \phi$ and, for homogeneous fluids, that is when $\rho$ is independent from space variables, to $w = \rho\, \phi + const.$. But a constant pressure don't give rise to any effect on the motion of the fluid, so the constant term can be ignored; this tell us the physical interpretation of $\varpi$: it is the impulsive pressure which would instantaneously generate from the rest the motion which actually exists (see also [76]).

The kinetic energy of the irrotational fluid is given by:

$$T = \frac{1}{2}\int \rho(\mathbf{q})^2 dV = \frac{1}{2}\int \rho(\operatorname{grad}\phi)^2 dV.$$

In the next lines it will be more useful to recast the previous expression of the kinetic energy in the following form:

$$T = \frac{\rho}{2}\int \phi\frac{\partial\phi}{\partial n}dS, \tag{C.2}$$

where $\frac{\partial}{\partial n}$ denotes the normal derivative to the surface ($\frac{\partial\phi}{\partial n} = \operatorname{grad}\phi\cdot\mathbf{n}$) and having assumed the incompressibility of the fluid. Let me prove (C.2). First of all note that, by an application of the Gauss' theorem, the following formula holds:

$$T = \frac{\rho}{2}\int \phi\frac{\partial\phi}{\partial n}dS - \frac{\rho}{2}\int \phi\left(\nabla^2\phi\right)dV. \tag{C.3}$$

In fact, the previous formula is a plain application of the Gauss' theorem:

$$\int_V \operatorname{div}\mathbf{F}dV = \int_S \mathbf{F}\cdot\mathbf{n}\,dS \tag{C.4}$$

by taking $\mathbf{F} = \phi\operatorname{grad}\phi$, and using $\operatorname{div}(\phi\operatorname{grad}\phi) = \phi\nabla^2\phi + (\operatorname{grad}\phi)(\operatorname{grad}\phi)$. The equation (C.2) simply follows by (C.3) by noting that, under the hypotheses made, $\nabla^2\phi = 0$. In fact the continuity equation for the mass of fluid within the closed surface $S$ reads:

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{q}) = 0.$$

If the fluid is incompressible then by the previous equation simply follows $\operatorname{div}\mathbf{q} = 0$. If, in addition, the fluid is irrotational, then $\mathbf{q} = -\operatorname{grad}\phi$, which, inserted in $\operatorname{div}\mathbf{q} = 0$, gives indeed $\nabla^2\phi = 0$.

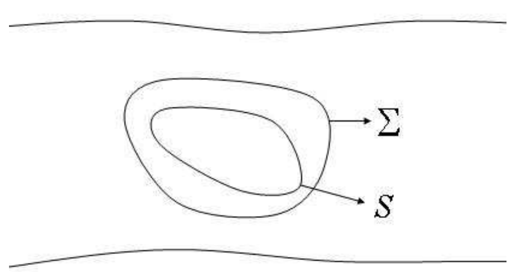


Figure C.1: The surface $\Sigma$ enclosing the solid $S$.

**The kinetic energy of an infinite fluid.** Formula (C.2) gives the kinetic energy of an irrotational, incompressible fluid enclosed in a finite surface. Now I will show how it is possible to extract, from this formula, the kinetic energy of an irrotational infinite incompressible fluid. At this aim let me take an infinite fluid, at rest at infinity, and bounded internally by a solid $S$. I also take a surface $\Sigma$ enclosing completely the solid $S$ as in figure (C.1). Consider the fluid occupying the region between the surface $\Sigma$ and the solid $S$. Since it is a finite region, one can apply the results of the previous paragraph, so that the kinetic energy of this fluid is given by:

$$T = \frac{\rho}{2}\left(\int_S \phi \frac{\partial \phi}{\partial n} dS + \int_\Sigma \phi \frac{\partial \phi}{\partial n} d\Sigma\right). \tag{C.5}$$

By the continuity equation

$$\frac{\partial}{\partial t}\int \rho dV = \rho\left(\int (\mathrm{grad}\,\phi \cdot \mathbf{n})\, dS + \int (\mathrm{grad}\,\phi \cdot \mathbf{n})\, d\Sigma\right) = 0, \tag{C.6}$$

so that it is possible to subtract any constant $C$ to $\phi$ in (C.5) and the equivalence remains true $\forall C$:

$$T = \frac{\rho}{2}\left(\int_S (\phi - C)\frac{\partial \phi}{\partial n} dS + \int_\Sigma (\phi - C)\frac{\partial \phi}{\partial n} d\Sigma\right). \tag{C.7}$$

Now, following [76], I will show that it is indeed possible to take the constant $C$ such that, in the limit $\Sigma \to \infty$, the integral $\int_\Sigma (\phi - C)\frac{\partial \phi}{\partial n} d\Sigma$ goes to zero. Furthermore, since there is no flow across $S$ for a solid boundary, the result for the kinetic energy is:

$$T = \frac{\rho}{2}\int \phi \frac{\partial \phi}{\partial n} dS, \tag{C.8}$$

where I recall that now $S$ is the surface of the solid immersed in the fluid. The reader satisfied of this assertion can skip the proof and pass to the next paragraph.
Taking $\Sigma$ as a sphere of radius $R$, then the versor normal to the surface is $d\mathbf{n} = d\mathbf{R}/R$ and the surface element corresponds to $d\Sigma = R^2 d\Omega$, being $d\Omega$ the solid angle subtended by $d\Sigma$. The continuity equation (C.6) gives:

$$R^2 \int \frac{\partial \phi}{\partial R} d\Omega = -\int_S \frac{\partial \phi}{\partial n}\, dS \doteq -F,$$

where by $F$ I have defined the total flow across $S$. If $S$ is the surface of a solid, obviously $F = 0$. The mean value of the velocity potential on the sphere $\Sigma$ is:

$$\begin{gathered}\langle \phi \rangle_\Sigma \doteq M = \frac{1}{4\pi R^2}\int \phi d\Sigma = \frac{1}{4\pi}\int \phi d\Omega \Longrightarrow \frac{\partial M}{\partial R} = \frac{1}{4\pi}\int \frac{\partial \phi}{\partial R} d\Omega \Longrightarrow \\ \Longrightarrow \frac{\partial M}{\partial R} = -\frac{F}{4\pi R^2} \Longrightarrow M = \frac{F}{4\pi R} + c,\end{gathered} \tag{C.9}$$

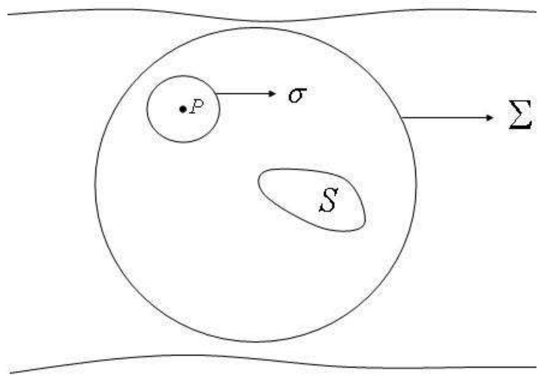


Figure C.2

where $c$ is independent of $R$. Note that $c$ can depend on the centre of the sphere, but in an infinite fluid $c$ is a true constant. In fact, considering $R$ fixed and displacing the centre of $\delta x$, then for $c$ one has $\frac{\partial c}{\partial x} = \frac{\partial M}{\partial x} = \frac{1}{4\pi}\int \frac{\partial \phi}{\partial x} d\Omega$ and taking $R \to \infty$, since $\frac{\partial \phi}{\partial x}$ vanish at infinity, $c$ is independent of $x$. In the same way it is also independent of $y$ and $z$. If the total flow across $S$ is equal to zero, then this constant is equal to $M$. Really, in this case not only the mean value of $\phi$ on $\Sigma$ is equal to $c$, but furthermore the stronger condition $\phi(P) \underset{P\to\infty}{\longrightarrow} C$ holds. This assertion can be proved with the help of Gauss' theorem. In fact, taking in (C.4) $\mathbf{F} = a \operatorname{grad} b$, where $a$ e $b$ are two arbitrary functions, and using the formula div $(a \operatorname{grad} b) = a\nabla^2 b + (\operatorname{grad} a)\cdot(\operatorname{grad} b)$, one easily finds:

$$\int (\operatorname{grad} a)\cdot(\operatorname{grad} b) dV = -\int a\nabla^2 b dV + \int a\frac{\partial b}{\partial n} dS = -\int b\nabla^2 b dV + \int b\frac{\partial a}{\partial n} dS.$$

Note that by the last equivalence it follows that, if $a$ and $b$ are two harmonic functions inside $S$, then:

$$\int \left( a\frac{\partial b}{\partial n} - b\frac{\partial a}{\partial n}\right) dS = 0. \tag{C.10}$$

This result can be used as follows. Take the surface enclosing the solid $S$ again as the sphere $\Sigma$ with centre $P$. Now enclose the centre $P$ with another little sphere $\sigma$, with the solid angle equal to $d\omega$, see figure (C.2). Let $r'$, $r$ and $R$ be respectively the distances of the point $P$ from a point on the surfaces $S$, $\sigma$ and $\Sigma$. Note that the velocity potential and the inverse of a distance are harmonic functions, so the application of (C.10) to the region between $S$, $\sigma$ and $\Sigma$ gives:

$$\begin{aligned}&\int_S \left(\phi\frac{\partial(1/r')}{\partial n} - \frac{1}{r'}\frac{\partial\phi}{\partial n}\right) dS - \int_\sigma \left(\phi\frac{\partial(1/r)}{\partial r} - \frac{1}{r}\frac{\partial\phi}{\partial r}\right) r^2 d\omega + \\ &+ \int_\Sigma \left(\phi\frac{\partial(1/R)}{\partial R} - \frac{1}{R}\frac{\partial\phi}{\partial R}\right) R^2 d\Omega = 0.\end{aligned}$$

Note also that the minus sign in front of the second integral is due to the convention that the positive direction of $\mathbf{n}$ is outside the surface, so that for $\sigma$ the versor $\mathbf{n}$ enters *inside* the sphere. If the surface $\sigma$ goes into the point $P$, then it is readily seen that the second integral gives $4\pi\phi(P)$. As for the the integral on $\Sigma$, one has:

$$-\int \phi d\Omega - R\int \frac{\partial\phi}{\partial R} d\Omega = -4\pi M + \frac{F}{R} = -4\pi c.$$

By putting all together and sending $P$ to infinity, i.e. $r' \to \infty$, one has the result $4\pi(c - \phi(P)) = 0$ when $P \to \infty$. Note that in the application of this result to the kinetic energy of an infinite fluid with a solid immersed in it, one has to take care that in the expression (C.2) the versor $\mathbf{n}$ is *inwards* the solid.

**The Kirchhoff equations** Consider a solid $S$ immersed in an infinite fluid, both at rest. If the solid be set in motion in any manner, the resulting motion of the fluid will be irrotational [76]. Since the kinetic energy of the fluid is finite, the velocity at infinity must be zero. The velocity potential then as to satisfy the conditions:

$$\nabla^2\phi = 0, \qquad \nabla\phi = 0 \quad \text{at infinity.} \tag{C.11}$$

Now consider the solid. In a frame of reference fixed relatively to the solid, its motion is specified by the velocity $\mathbf{u}$ of the origin, and by the angular velocity $\boldsymbol{\omega}$. The point $\mathbf{r}$ on the boundary of the solid possesses the velocity $\mathbf{v} = \mathbf{u} + \boldsymbol{\omega} \wedge \mathbf{r}$, where $\wedge$ is the cross product. In terms of the velocity potential, since $\mathbf{v} = -\text{grad}\,\phi$, one has:

$$\frac{\partial\phi}{\partial n} = -\mathbf{n}\cdot(\mathbf{u} + \boldsymbol{\omega}\wedge\mathbf{r}). \tag{C.12}$$

Setting:

$$\phi = \mathbf{u}\cdot\boldsymbol{\varrho} + \boldsymbol{\omega}\cdot\boldsymbol{\chi}$$

and introducing this formula in (C.12):

$$\frac{\partial\boldsymbol{\varrho}}{\partial n} = -\mathbf{n}, \qquad \frac{\partial\boldsymbol{\chi}}{\partial n} = -\mathbf{r}\wedge\mathbf{n}.$$

This tell us that $\boldsymbol{\varrho}$ and $\boldsymbol{\chi}$ depends only on the geometry of the solid. Obviously now the conditions (C.11) must apply separately at all the components of both $\boldsymbol{\varrho}$ and $\boldsymbol{\chi}$. Having the conditions that $\phi$ has to satisfy at infinity and on the boundary of the solid, one can pass to analyse the kinetic energy of the solid $T_s$. The total kinetic energy will be given by adding that of the fluid with that of the solid.
The kinetic energy of the solid is given by:

$$T_S = \frac{1}{2}\int_V \sigma\,(\mathbf{u} + \boldsymbol{\omega}\wedge\mathbf{r})^2\, dV, \tag{C.13}$$

where obviously the integration is throughout the volume of the solid. This expression is a quadratic homogeneous function of $\mathbf{u}$ and $\boldsymbol{\omega}$, so by the Euler's theorem on homogeneous functions it follows:

$$\mathbf{u}\frac{\partial T_S}{\partial \mathbf{u}} + \boldsymbol{\omega}\frac{\partial T_S}{\partial \boldsymbol{\omega}} = 2T_S.$$

From (C.13) one get at once:

$$\begin{aligned} \frac{\partial T_S}{\partial \mathbf{u}} &= \int_V \sigma\left(\mathbf{u} + \boldsymbol{\omega}\wedge\mathbf{r}\right) dV, \\ \frac{\partial T_S}{\partial \boldsymbol{\omega}} &= \int_V \sigma\mathbf{r}\wedge\left(\mathbf{u} + \boldsymbol{\omega}\wedge\mathbf{r}\right) dV, \end{aligned} \tag{C.14}$$

so that $\frac{\partial T_S}{\partial \mathbf{u}}$ and $\frac{\partial T_S}{\partial \boldsymbol{\omega}}$ are respectively the linear momentum and angular momentum of the solid.

Now pass to the kinetic energy of the fluid. For what we have shown in the previous paragraph, it is possible to write:

$$T_L = \frac{\rho}{2}\int_S \phi\frac{\partial\phi}{\partial n} dS = -\frac{\rho}{2}\int_S \left(\mathbf{u}\cdot\boldsymbol{\varrho} + \boldsymbol{\omega}\cdot\boldsymbol{\chi}\right)\left(\mathbf{n}\cdot\left(\mathbf{u} + \boldsymbol{\omega}\wedge\mathbf{r}\right)\right).$$

So, again, by the Euler's theorem on homogeneous functions:

$$\mathbf{u}\frac{\partial T_L}{\partial \mathbf{u}} + \boldsymbol{\omega}\frac{\partial T_L}{\partial \boldsymbol{\omega}} = 2T_L.$$

Let me calculate $\frac{\partial T_L}{\partial \mathbf{u}}$ and $\frac{\partial T_L}{\partial \boldsymbol{\omega}}$.

$$\begin{aligned} \frac{\partial T_L}{\partial \mathbf{u}} =& \frac{\rho}{2}\int_S \frac{\partial}{\partial \mathbf{u}}\left(\phi\frac{\partial\phi}{\partial n}\right) dS = \frac{\rho}{2}\int_S \left(\left(\frac{\partial\phi}{\partial \mathbf{u}}\right)\left(\frac{\partial\phi}{\partial n}\right) + \phi\frac{\partial}{\partial \mathbf{u}}\left(\frac{\partial\phi}{\partial n}\right)\right) dS = \\ & \frac{\rho}{2}\int_S \left(\boldsymbol{\varrho}\frac{\partial\phi}{\partial n} - \mathbf{n}\phi\right) dS. \end{aligned}$$

Applying the formula (C.10) for $a = \phi$ and $\mathbf{b} = \boldsymbol{\varrho}$ (obviously (C.10) is valid for any component of $\mathbf{b}$ and so is valid for all the vector $\mathbf{b}$):

$$\int_S \boldsymbol{\varrho}\frac{\partial\phi}{\partial n} dS = \int_S \phi\frac{\partial\boldsymbol{\varrho}}{\partial n} dS = -\int_S \phi\mathbf{n} dS,$$

so, inserting this result in the formula for $\frac{\partial T_L}{\partial \mathbf{u}}$:

$$\frac{\partial T_L}{\partial \mathbf{u}} = -\rho\int_S \phi\mathbf{n} dS. \tag{C.15}$$

Proceeding in the same way for $\frac{\partial T_L}{\partial \boldsymbol{\omega}}$, one obtains:

$$\frac{\partial T_L}{\partial \boldsymbol{\omega}} = -\rho \int_S \phi(\mathbf{r} \wedge \mathbf{n}) dS. \tag{C.16}$$

These expressions are the linear momentum and impulsive moment applied by the solid to the boundary of the fluid in contact with it. So the total impulse and its vector momentum are given, for the system $solid + fluid$ respectively by $\frac{\partial T_S}{\partial \mathbf{u}} + \frac{\partial T_L}{\partial \mathbf{u}}$ and $\frac{\partial T_S}{\partial \boldsymbol{\omega}} + \frac{\partial T_L}{\partial \boldsymbol{\omega}}$.

Take now a frame of references fixed in the solid and consider the change in the total linear momentum $\mathbf{p}$ and its vector momentum $\mathbf{J}$. In a short interval $dt$, the solid is rotated by an angle $\boldsymbol{\omega} dt$ and is translated of a quantity $\mathbf{u}dt$. Consider the effect of translation and rotation on the change of $\mathbf{p}$ and $\mathbf{J}$ estimated by an observer moving with the solid. The translation obviously does not change the linear momentum $\mathbf{p}$, but the lever arm changes of $\mathbf{u}dt$, so that $\mathbf{J} \to \mathbf{J} + \mathbf{u}dt \wedge \mathbf{p}$. The effect of the rotation is instead to change a vector $\mathbf{v}$ into $\mathbf{v} + \boldsymbol{\omega} dt \wedge \mathbf{v}$, so that $\mathbf{p} \to \mathbf{p} + \boldsymbol{\omega} dt \wedge \mathbf{p}$ and $\mathbf{J} \to \mathbf{J} + \boldsymbol{\omega} dt \wedge \mathbf{J}$. At this variations it must be added the variation in time of $\mathbf{p}$ and $\mathbf{J}$ as seen in the frame moving with the solid, say $\frac{d\mathbf{p}}{dt}$ and $\frac{d\mathbf{J}}{dt}$. Summarizing, since the total variation of $\mathbf{p}$ and $\mathbf{J}$ must equate the total external force $\mathbf{F}$ on the system and its momentum $\mathbf{M}$, putting all together:

$$\begin{cases} \dfrac{d\mathbf{p}}{dt} + \boldsymbol{\omega} \wedge \mathbf{p} = \mathbf{F}, \\ \dfrac{d\mathbf{J}}{dt} + \boldsymbol{\omega} \wedge \mathbf{J} + \mathbf{u} \wedge \mathbf{p} = \mathbf{M}. \end{cases}$$

In our case, since $\mathbf{p} = \frac{\partial T}{\partial \mathbf{u}} = \frac{\partial T_S}{\partial \mathbf{u}} + \frac{\partial T_L}{\partial \mathbf{u}}$ and $\mathbf{J} = \frac{\partial T}{\partial \boldsymbol{\omega}} = \frac{\partial T_S}{\partial \boldsymbol{\omega}} + \frac{\partial T_L}{\partial \boldsymbol{\omega}}$, one obtains the Kirchhoff equations:

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial T}{\partial \mathbf{u}}\right) + \boldsymbol{\omega} \wedge \frac{\partial T}{\partial \mathbf{u}} = \mathbf{F}, \\ &\frac{d}{dt}\left(\frac{\partial T}{\partial \boldsymbol{\omega}}\right) + \boldsymbol{\omega} \wedge \frac{\partial T}{\partial \boldsymbol{\omega}} + \mathbf{u} \wedge \frac{\partial T}{\partial \mathbf{u}} = \mathbf{M}. \end{aligned} \tag{C.17}$$

Note that, simply moving on the r.h.s. the terms proportional to $T_L$, it becomes evident that the net effect of the fluid pressure on the solid is represented by a force $\mathbf{F}_L$ and a couple $\mathbf{L}_L$ given by:

$$\begin{aligned} \mathbf{F}_L &= -\frac{d}{dt}\left(\frac{\partial T_L}{\partial \mathbf{u}}\right) - \boldsymbol{\omega} \wedge \frac{\partial T_L}{\partial \mathbf{u}}, \\ \mathbf{L}_L &= -\frac{d}{dt}\left(\frac{\partial T_L}{\partial \boldsymbol{\omega}}\right) - \boldsymbol{\omega} \wedge \frac{\partial T_L}{\partial \boldsymbol{\omega}} - \mathbf{u} \wedge \frac{\partial T_L}{\partial \mathbf{u}}. \end{aligned}$$

Now let me give some general considerations about the kinetic energy. As I have shown, the total kinetic energy of the system *solid* + *fluid* is given by an expression of the type:

$$T = \frac{1}{2}\sum_{i\geq j}(A_{ij}u_iu_j + B_{ij}\omega_i\omega_j) + \frac{1}{2}\sum_{i,j}C_{ij}u_i\omega_j, \tag{C.18}$$

where the quantities $A_{ij}, B_{ij}, C_{ij}$ depends only on the shape of the solid. If the solid has three perpendicular planes of symmetry (ellipsoids, parallelepipeds ...), changing the sign of the velocities does not affect the amount of kinetic energy, so that $C_{ij} = 0$ and only the diagonal elements in $A_{ij}$ and $B_{ij}$ survive. In this case, Clebsch [30] discovered that the dynamical system governed by the Hamiltonian:

$$T = \frac{1}{2}\left(A_{11}u_1^2 + A_{22}u_2^2 + A_{33}u_3^2 + (B_{11}\omega_1^2 + B_{22}\omega_2^2 + B_{33}\omega_3^2\right), \tag{C.19}$$

with the following constraint on the quantities $A_{ii}$ and $B_{ii}$:

$$\frac{A_{11} - A_{22}}{B_{33}} + \frac{A_{22} - A_{33}}{B_{11}} + \frac{A_{33} - A_{11}}{B_{22}} = 0,$$

is integrable. If in addition the solid is of revolution, say around the $z$ axis, or is a right prism whose section is any regular polygon (see [69]), then $A_{11} = A_{22}$ and $B_{11} = B_{22}$ and the total kinetic energy becomes (with $A_{ii} \doteq A_i, \quad B_{ii} \doteq B_i$):

$$T = \frac{1}{2}\Big(A_1(u_1^2 + u_2^2) + A_3u_3^2\Big) + \frac{1}{2}\Big(B_1(\omega_1^2 + \omega_2^2) + B_3\omega_3^2\Big). \tag{C.20}$$

The total impulse $\mathbf{p}$ and angular momentum $\mathbf{J}$ of the system are given by:

$$p_i = \frac{\partial T}{\partial u_i}, \qquad J_i = \frac{\partial T}{\partial \omega_i}, \tag{C.21}$$

so that in terms of $\mathbf{p}$ and $\mathbf{J}$ the kinetic energy (C.20) can be rewritten as in (4.1):

$$T = \frac{1}{2}\Big(\frac{p_1^2 + p_2^2}{A_1} + \frac{p_3^2}{A_3}\Big) + \frac{1}{2}\Big(\frac{J_1^2 + J_2^2}{B_1} + \frac{J_3^2}{B_3}\Big). \tag{C.22}$$

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